ANALYSIS OF FRICTIONAL CONTACT MODELS FOR DYNAMIC SIMULATION *

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Abstract

Simulation of dynamic systems possessing unilateral frictional contacts is important to many industrial applications. While rigid body models are often employed, it is well established that friction can cause problems with the existence and uniqueness of the forward dynamics problem. In these situations, we argue that compliant contact models, while increasing the length of the state vector, successfully resolve these ambiguities. The simplicity and efficiency of rigid body models, however, provide strong motivation for their use during those portions of a simulation when the compliant contact model indicates a unique and stable solution. We use singular perturbation theory in combination with linear complementarity theory to establish conditions for the validity of the rigid body model with rolling and sliding unilateral contacts for planar systems. The results are illustrated with a simple example.

1 Introduction

Rigid body dynamic simulation There are many applications in an industrial setting where it is beneficial to understand the dynamics of systems with frictional contacts. Examples include part-feeding systems [1] and automatic assembly of mechanical components [2]. When a component is fed, typically at high speeds, along guides or rollers, it may experience multiple frictional contacts with surrounding rigid bodies before arriving at its final destination. Similarly, during the insertion of a peg into a hole, there may be several contacts between the peg and the hole before successful assembly [3]. Examples of robotic systems with frictional contacts include multifingered grippers [4], multiarm manipulation systems, legged locomotion systems, and wheeled robots on uneven terrain. Finally, there are many examples of mechanical systems in which frictional contacts are essential to the successful operation of the system [5]. In order to successfully design and optimize such mechanical systems or manufacturing processes, a method for modeling and simulating mechanical systems with frictional contacts is necessary [6].

Systems with frictional contacts The dynamic equations of motion for a mechanical system comprised of rigid bodies can be written in the form:

$$M\ddot{q} + c(q,\dot{q}) = \tau + \Phi_q^T \lambda \tag{1}$$

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where $q \in \Re^n$ is the vector of generalized coordinates, M is an $n \times n$ positive-definite symmetric inertia matrix, $c(q,\dot{q})$ is a $n \times 1$ vector of nonlinear inertial forces, τ is the vector of applied (external) forces and torques, and λ is the vector of constraint forces. The system is subject to p constraints:

$$\Phi(q) = [\phi_1(q), \cdots, \phi_p(q)]^T \geq 0 \tag{2}$$

and Φ_q in Equation (1) is the $p \times n$ Jacobian matrix, $\frac{\partial \Phi}{\partial q}$. We will assume, without loss of generality, that this does not include bilateral, holonomic constraints. Further, for the sake of simplicity, we will assume that nonholonomic constraints are not present. Let there be m contacts, consisting of r rolling contacts and s sliding contacts. Let the subscripts N and T denote quantities in the normal and tangential directions and the subscripts S and R denote sliding and rolling (sticking) contacts respectively. The Jacobian matrix and constraint forces in (1) for a planar system are given by:

$$\begin{aligned} \Phi_q^T &= \left[\begin{array}{ccc} \hat{\Phi}_{S_q}^T & \Phi_{NR_q}^T & \Phi_{TR_q}^T \end{array} \right], \\ \hat{\Phi}_{S_q} &= \left[\Phi_{NS_q} - \Phi_{TS_q} diag(\mu_s sign(\dot{\Phi}_{TS_q})) \right], \\ \lambda &= \left[\lambda_{NS}^T \; \lambda_{NR}^T \; \lambda_{TR}^T \right]^T, \end{aligned} \tag{3}$$

where $p = m + r$ and Coulomb's law is employed.

Contacts between rigid bodies generate complementary constraints on the position (or velocity or acceleration) variables and the corresponding force variables as detailed in [7, 8]. These conditions allow active contacts to become inactive. The case of inactive contacts becoming active is modeled by rigid body impacts and is treated elsewhere [9, 10].

Existence and uniqueness The problem of determining contact forces for the rigid body problem can be reduced to a linear complimentarity problem (LCP) in the planar case [7, 8]. It is well-known that in the frictionless case, there is always a unique solution for $\ddot{q}$. When the constraints are not all independent (the rows of Φ_q are not linearly independent), the system is statically indeterminate and the constraint forces λ cannot be uniquely determined. In the frictional case, if all contacts are known to be rolling (sticking), then the relative tangential velocity, $\dot{\phi}_{T,i}$, is zero at each contact, and the existence of a solution can be shown if the Jacobian Φ_q is full rank [8]. In all other cases, the initial value problem can be shown to have no solution or multiple solutions for special choices of initial conditions.

Since the difficulties of proving existence and uniqueness arise due to the presence of unknown contact forces (λ) that are subject to nonlinear constraints, it is attractive to pursue models in which the contact forces are explicit functions of the state variables $(q, \dot{q})$. A continuum model for modeling the deformations at each contact is described in [11]. This general approach is further refined by [12]. Existence and uniqueness is shown for the special case in which the maximum tangential force at each point is *a priori* known. The disadvantage in this approach is the complexity of the model. The contact models lead to a high-dimensional, stiff system of equations and a run time that is unacceptable for real-time simulation.

The goal of the paper We develop a simplified model of compliance that overcomes the shortcomings of the rigid body model and successfully approximates the dynamics of the continuum model. The second goal of the paper is to examine the *stability* of the solutions obtained by the rigid body model. When the reduced order rigid body model and the more complex compliant contact model agree in their predictions, it is attractive to pursue the simulation using rigid body models. We argue that rigid body dynamic simulation is meaningful only when the solution of the compliant contact model converges to the solution of the rigid body model. Finally, we use methods from singular perturbation analysis to establish conditions under which the solution predicted by the rigid body model is stable.

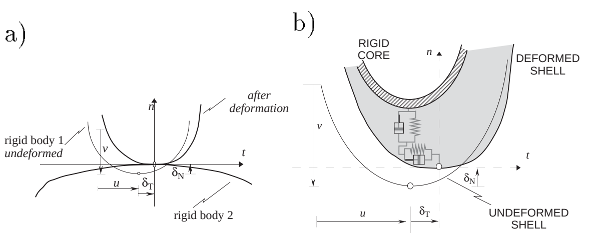

Figure 1: a) A simple model of contact compliance. Each body consists of a rigid core and a small compliant shell, the deformations of which are modeled by (possibly nonlinear) springs and dampers. b) A viscoelastic model for contact compliance

2 Compliant Contact Models

In this section, we develop a formulation that incorporates a local model of compliance at each contact. This formalism assumes that the principles of rigid body dynamics are valid and the gross motion of the dynamic system is described by the state variables $(q, \dot{q})$. However, in addition to the gross motion, there are small (local) deformations at each contact. To see this, consider a rigid body surrounded by a very thin deformable shell the inertia of which is negligible, as shown in the schematic in Figure 1. The thickness of the deformable shell is negligible compared to the radius of curvature of the rigid body. In the planar case, the gross rigid body motion $(q, \dot{q})$ determines the nominal displacement of the contact point $(u,\, v)$. The actual displacement of the contact point is given by $(u + \delta_T,\, v + \delta_N)$. The contact forces, on the other hand, are related to the normal and tangential deformations (δ_N, δ_T) of the shell and their derivatives $(\dot{\delta}_N, \dot{\delta}_T)$. Since the deformations can be determined by the knowledge of the material properties of the deformable shell, the contact forces can be related to the original state variables $(q, \dot{q})$.

It is important to note that the compliance in the tangential direction allows small displacements at a sticking contact in the tangential direction before slip actually occurs[1]. This can be seen if we press a finger against a rigid surface and attempt to move it tangential to the surface. There will be a small displacement before gross rigid body sliding motion begins. This is in contrast to what a Coulomb-like friction law, in conjunction with conventional rigid body models, predicts — the tangential force is a discontinuous function of velocity. The tangential compliance makes the tangential force a continuous (in fact, piecewise smooth) function of tangential velocity and provides for a smooth (continuous) transition between the sticking and the sliding regimes.

A general viscoelastic model for contact compliance is shown in Figure 1. At contact i, the normal and tangential contact forces ($\lambda_{N,i}$ and $\lambda_{T,i}$) between the two contacting bodies may be modeled as:

$$\lambda_{N,i} = f_{N,i}\left(\delta_{N,i}\right) + g_{N,i}\left(\delta_{N,i}, \dot{\delta}_{N,i}\right) \qquad (4)$$

$$\lambda_{T,i} = f_{T,i}\left(\delta_{T,i}\right) + g_{T,i}\left(\delta_{T,i}, \dot{\delta}_{T,i}\right) \qquad (5)$$

where the functions $f_{N,i}$ and $f_{T,i}$ are the elastic stiffness terms and $g_{N,i}$ and $g_{T,i}$ are the damping terms in the normal and tangential directions respectively. These functions depend on the geometry and material properties of the two bodies in contact and may be nonlinear. While the constitutive laws relating the forces to the deformations (and their derivatives) may be quite general, it is important to note that we have decoupled the modeling of the contact forces (*i.e.* the force at a contact is only dependent on the deformation at that contact) and further the expressions for the normal and tangential forces are uncoupled. The details of the use of the compliant model solution procedure can be found in [9].

There are two disadvantages of the compliant contact model. First it is clear that we now need to model the contacts and this increases the possibility of modeling errors, particularly since contact models are notoriously difficult to obtain. Second, and more importantly from an algorithmic and mathematical standpoint, there is a need to extend the state space of the system. The original rigid body model has $n - (m + r)$ degrees of freedom, and a state space that is $2n - 2(m + r)$-dimensional. In the compliant contact model the state consists of $(q, \dot{q})$ and is $2n$-dimensional, regardless of the number of contacts. The three main advantages are:

- The normal and tangential forces are now uniquely determined and there is no question of static indeterminacy;
- The difficulties with uniqueness and existence no longer arise; and

[1]This is similar to the phenomena of microslip that is described in the tribology and contact mechanics literature [13].

- As shown in [9], a model with tangential contact compliance is more realistic and can better explain such physical observations as microslip and hysteresis.

We do not wish to promote unnecessary model complexity, however, and in those situations when a compliant contact model is not needed, it would be desirable to retain the simpler rigid body model. The popularity of rigid body models can be attributed not only to their simplicity, but also to the fact that they have produced adequate results in a broad range of applications. Clearly, rigid body models can only be used when a unique solution can be determined without any additional ad hoc assumptions. In the next section, we will use singular perturbation theory to investigate the *stability* of the solutions obtained from the rigid body model. If a rigid body solution is stable, the compliant contact model solution converges to this solution and the compliance can be neglected. However, if this solution is unstable, one must use the more sophisticated compliant contact model for simulation.

3 Singular Perturbation Analysis

The rigid body model leads to a set of differential-algebraic equations (DAEs) as shown in Section 1. We argued in the previous section that a compliant contact model that explicitly models the small deformations is a more accurate model. These deformations are an order of magnitude smaller than the gross motions of the mechanical system. By setting these deformations to zero (or by allowing the corresponding stiffnesses to be infinitely large), we recover the DAEs of the rigid body model.

This suggests (see [14] for a more lucid exposition) that we might be able to invent a small perturbation parameter $\epsilon \ll 1$ that enters the state equations for the compliant contact model, such that, by setting $\epsilon = 0$ the state equations degenerate into the DAEs of Section 1. Further, there are two time scales in the dynamics of the mechanical system [14, 15]. The first time scale, is the one that corresponds to the reduced order rigid body model dynamics. The second time scale is the fast time scale that characterizes the contact dynamics.

Singular perturbation theory allows us to decompose the system model into reduced (slow) and boundary-layer (fast) models [16]. The response of the system, described by the compliant model, consists of the slow response and the fast transient. If the boundary-layer model is exponentially stable, the fast transients will exponentially converge to zero and it is reasonable to neglect the high-frequency contact dynamics. In such a situation, the reduced order model obtained by neglecting the compliance (and the effects of the small parameter ϵ), is robust to the unmodeled dynamics. If the boundary-layer model is not stable, we cannot neglect these terms and it is necessary to use the complete dynamic model incorporating compliance.

We will basically follow the approach of [14] in developing the reduced (slow) and boundary-layer (fast) models for the mechanical system (1). In a previous paper [15], one of the authors developed a framework for analyzing the effect of compliance in the normal direction on systems with sliding contact and derived conditions under which this compliance could be neglected. Here we will pursue a unified approach to the analysis of rolling and sliding contacts, and establish conditions under which the details of the compliant contact model can be neglected. We will use the simple Kelvin-Voigt model [14] to model contacts and assume, for the sake of simplicity, all contacts are identical. However, our approach is quite general, and as such, works for any constitutive law that has the form (4-5).

We use x to denote the slow variables and ϵz to denote the fast variables, with $\epsilon \ll 1$. ϵz corresponds to the small deformations, that is, small deviations from the rigid body constraints. $q_2 \in \Re^{n-(m+r)}$ is a subset of the vector of generalized coordinates which is partitioned as $q = \left[q_1^T q_2^T\right]^T$ so that

$$\begin{bmatrix} \epsilon z \\ x \end{bmatrix} = \begin{bmatrix} \Phi_N(q_1, q_2) \\ \Phi_{TR}(q_1, q_2) \\ q_2 \end{bmatrix}$$

is a valid choice of coordinates. In order for this to be true, the implicit function theorem requires that the Jacobian matrix

$$\Gamma = \begin{bmatrix} \Phi_{Nq(m\times n)} \\ \Phi_{TRq(r\times n)} \\ 0_{((n-(m+r))\times(m+n))} \quad I_{((n-(m+r))\times(n-(m+r)))} \end{bmatrix} \tag{6}$$

be nonsingular. This in turn implies

$$\begin{bmatrix} \Phi_{Nq} \\ \Phi_{TRq} \end{bmatrix} \epsilon \Re^{(m+r)\times n}$$

must be full rank. If these conditions are satisfied, we may write:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J(\epsilon z, x) \begin{bmatrix} \epsilon \dot{z} \\ \dot{x} \end{bmatrix}$$

where $J = \Gamma^{-1}$, and Equation (1) can be written in the form:

$$\begin{bmatrix} \epsilon \ddot{z} \\ \ddot{x} \end{bmatrix} = -J^{-1}\dot{J} \begin{bmatrix} \epsilon \dot{z} \\ \dot{x} \end{bmatrix} + J^{-1}M^{-1}(\tau - c + \left[\hat{\Phi}_{S_q}^T \ \Phi_{NRq}^T \ \Phi_{TRq}^T\right]\lambda) \tag{7}$$

With the Kelvin-Voigt model, the normal and tangential contact forces take the form:

$$\lambda = -(\frac{b}{\sqrt{\epsilon}}\epsilon \dot{z} + \frac{k}{\epsilon}\epsilon z) \tag{8}$$

where $\frac{k}{\epsilon}$ and $\frac{b}{\sqrt{\epsilon}}$ are the stiffness and damping constants respectively. The scaling of these constants is obtained through dimensional analysis (for example, see [14, 15]). The first $m + r$ equations model the fast dynamics (boundary layer) and the remaining $n - (m + r)$ represent the slow dynamics.

We let $\epsilon \to 0$ in the transformed equations of motion and solve for the steady state solution, $\overline{z}$, from the first $m + r$ equations. $\overline{z}$ is then substituted in to the last $n - (m + r)$ equations to obtain the slow dynamics. Note that the $\overline{z}_i$ corresponding to the normal contact constraints must be negative while the $\overline{z}_i$ corresponding to the tangential rolling constraints have no restriction on sign.

To proceed with the stability analysis, we perform a time scale transformation $\tau = \frac{t}{\sqrt{\epsilon}}$ on the fast dynamics equations along with a change of variables $z = y + \overline{z}$ to move the quasi-steady state of z to the origin and arrive at the homogeneous boundary layer dynamics of the form:

$$y'' + bDy' + kDy = 0 \tag{9}$$

where $'$ denotes differentiation with respect to τ and

$$\mathrm{D}=\begin{bmatrix} \Phi_{Nq} \\ \Phi_{TRq} \end{bmatrix} \mathrm{M}^{-1} \begin{bmatrix} \hat{\Phi}_{S_q}^T & \Phi_{NRq}^T & \Phi_{TRq}^T \end{bmatrix} \tag{10}$$

These equations represent the transient of z. The stability of these equations depends on the eigenvalues of D as well as the value of $\frac{b^2}{k}$.

4 The One Contact Case

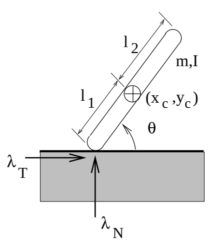

Figure 2: Planar rigid rod in contact with horizontal surface.

Consider the classic problem of the planar sliding rod. A rigid, slender rod of length $l = l_1 + l_2$ is in contact with a rigid horizontal surface, where l_1 is the distance from the contact to the C.M. and l_2 is the distance from the C.M. to the non-contacting end of the rod. The rod has mass m and centroidal moment of inertia I. μ is the coefficient of friction between the rod and surface. $q = [x_c \; y_c \; \theta]^T$ represent the generalized coordinates for the rod which are the position of the C.M. and the angular orientation. [2]

The impenetrability constraint of the tip of the rod in contact is: $\phi_N = y_c - l_1 s\theta \geq 0$. The gradient of the normal and tangential constraints at the contact point are:

$$\begin{aligned} \phi_{Nq} &= [0 \; 1 \; -l_1 c\theta] & (11) \\ \phi_{Tq} &= [1 \; 0 \; \; l_1 s\theta] & (12) \end{aligned}$$

and the relative normal acceleration at the contact, η_N, is given by:

$$\eta_N = (\phi_{Nq}\dot{q})_q\dot{q} + \phi_{Nq}M^{-1}(\tau - c) + \phi_{Nq}M^{-1}[\phi_{Nq}^T \; \phi_{Tq}^T]\begin{bmatrix} \lambda_N \\ \lambda_T \end{bmatrix}$$

[2] $s\theta$ and $c\theta$ denote $\sin\theta$ and $\cos\theta$ respectively.

4.1 With contact sliding

If we assume that the contact is sliding ($\dot{\phi}_N \neq 0$), the rigid body problem takes the form of an LCP.

$$\eta_N = b + A\lambda_N \geq 0, \; \lambda_N \geq 0, \; \eta_N\lambda_N = 0 \tag{13}$$

where

$$\begin{aligned} A &= \frac{1}{m} + \frac{l_1^2 c\theta}{I}(c\theta + \mu sgn(\dot{\phi}_T)s\theta) \\ b &= l_1\dot{\theta}^2 s\theta + \begin{bmatrix} 0 & \frac{1}{m} & -\frac{l_1 c\theta}{I} \end{bmatrix} \tau \end{aligned}$$

Note with $\mu = 0$, $A > 0$ and therefore A is a 1×1 P-matrix, which guarantees a unique solution. If $b \geq 0$ we are guaranteed of the existence of *a* solution regardless of whether or not A is a P-matrix.

If we now wish to pursue the singular perturbation approach to the problem, a transformation to a system of fast and slow variables can be accomplished by making the change of variables:

$$\begin{bmatrix} \epsilon\dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\phi}_N \\ \dot{x}_c \\ \dot{\theta} \end{bmatrix} = J^{-1}\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} = J^{-1}\dot{q}$$

The inverse function theorem requires:

$$det(J^{-1}) = det\begin{bmatrix} 0 & 1 & -l_1 c\theta \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 \neq 0$$

So we can always transform to the system of fast and slow variables and back.

Using the Kelvin-Voigt contact force model in the fast dynamics equation, letting $\epsilon \rightarrow 0$ and solving for z gives:

$$\overline{z} = \frac{b}{k_N A} \tag{14}$$

In order to satisfy the constraints of non-negativity of the normal contact force and normal relative separation of the rigid body model (achieved as $\epsilon \rightarrow 0$), the value for $\overline{z}$ in the above expression must be < 0 (A and b of different signs).

With time scale transformation and change of coordinates, we arrive at boundary layer dynamics of the form of equation (9). For a system with only sliding (or frictionless) contacts the matrix in the boundary layer dynamics, D, equals the matrix in the rigid body LCP formulation, A.

Since we assume $k_N, b_N > 0$, the stability of the boundary layer depends on the eigenvalues of the matrix:

$$D = A = \frac{1}{m} + \frac{l_1^2 c\theta}{I}(c\theta + \mu sgn(\dot{\phi}_T)s\theta)$$

As noted in [15], for the case of the planar rod with one sliding contact, the condition for stability of the boundary layer is identical to the requirement for the matrix to be a P-matrix in the LCP formulation and thus ensure the existence of a unique solution for arbitrary input. Based on the value of $\overline{z}$, the singular perturbation analysis may be used to test the stability in those situations where LCP analysis tells us that the contact is maintained. A summary of the results is given in Table 1.

Table 1: LCP and stability results for 1 sliding contact (C=contact, NC=no contact, NS=no solution, * denotes a case where the rod skims over the surface without generating contact forces)

Conditions		Solutions	Stability
$A > 0$	$b > 0$	NC	stable
$A > 0$	$b = 0$	NC^*	stable
$A > 0$	$b < 0$	C	stable
$A = 0$	$b > 0$	NC	stable
$A = 0$	$b = 0$	∞ solns.	-
$A = 0$	$b < 0$	NS	N/A
$A < 0$	$b > 0$	C	unstable
		NC	stable
$A < 0$	$b = 0$	NC^*	stable
$A < 0$	$b < 0$	NS	N/A

4.2 With contact rolling

If we now assume the contact is rolling or sticking (*i.e.* no relative tangential motion at the contact) with the definition of surplus and slack variables, the planar rod problem with a rolling (sticking) contact can again be formulated as an LCP [8].

$$s_T^+ = \mu\lambda_N + \lambda_T,\ s_T^- = \mu\lambda_N - \lambda_T,\ s_T^+, s_T^- \geq 0 \quad (15)$$

$$\lambda_N \geq 0,\ \eta_N \geq 0,\ \eta_N\lambda_N = 0 \quad (16)$$

$$\eta_T = \eta_T^+ - \eta_T^-,\ \eta_T^+ s_T^+ = 0,\ \eta_T^- s_T^- = 0 \quad (17)$$

$$\begin{bmatrix} \eta_N \\ \eta_T^+ \\ s_T^- \end{bmatrix} = \begin{bmatrix} \dot{\phi}_{Nq}\dot{q} + \phi_{Nq}M^{-1}(\tau - c) \\ \dot{\phi}_{Tq}\dot{q} + \phi_{Tq}M^{-1}(\tau - c) \\ 0 \end{bmatrix} \quad (18)$$

$$+ \begin{bmatrix} \phi_{Nq}M^{-1}(\phi_{Nq}^T - \phi_{Tq}^T\mu) & \phi_{Nq}M^{-1}\phi_{Tq}^T & 0 \\ \phi_{Tq}M^{-1}(\phi_{Nq}^T - \phi_{Tq}^T\mu) & \phi_{Tq}M^{-1}\phi_{Tq}^T & 1 \\ 2\mu & -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_N \\ s_T^+ \\ \eta_T^- \end{bmatrix}$$

The coefficient matrix in the LCP problem is

$$A = \begin{bmatrix} \frac{1}{m} + \frac{l_1^2}{I}c\theta(c\theta + \mu s\theta) & -\frac{l_1^2 c\theta s\theta}{I} & 0 \\ -\frac{\mu}{m} - \frac{l_1^2}{I}s\theta(c\theta + \mu s\theta) & \frac{1}{m} + \frac{l_1^2 s^2\theta}{I} & 1 \\ 2\mu & -1 & 0 \end{bmatrix}. \quad (19)$$

This matrix is never a P-matrix.

Again we pursue a singular perturbation approach to the problem and model the static frictional force with the compliant contact model as was done for the normal force. To perform the reversible change of variables,

$$\begin{bmatrix} \epsilon\dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\phi}_N \\ \dot{\phi}_T \\ \dot{\theta} \end{bmatrix} = J^{-1}\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} = J^{-1}\dot{q}$$

the inverse function theorem requires:

$$\Gamma = J^{-1} = \begin{bmatrix} 0 & 1 & -l_1 c\theta \\ 1 & 0 & l_1 s\theta \\ 0 & 0 & 1 \end{bmatrix}$$

be nonsingular. Since $|\Gamma| = -1$, this transformation is always possible. After changing variables and letting $\epsilon \to 0$, solving for z in the two fast equations gives:

$$\begin{bmatrix} \overline{z}_1 \\ \overline{z}_2 \end{bmatrix} = ml_1\dot{x}^2 \begin{bmatrix} \frac{Sx}{k_N} \\ \frac{Cx}{k_T} \end{bmatrix} \quad (20)$$

$$+ \begin{bmatrix} \frac{1}{k_N} & 0 \\ 0 & \frac{1}{k_T} \end{bmatrix} \begin{bmatrix} \frac{l_1^2 CxSx}{I} & \frac{1}{m} + \frac{l_1^2 S^2x}{I} & -\frac{l_1 Cx}{I} \\ \frac{1}{m} + \frac{l_1^2 C^2x}{I} & \frac{l_1^2 CxSx}{I} & \frac{l_1 Sx}{I} \end{bmatrix} \frac{\tau}{(\frac{1}{m} + \frac{l_1^2}{I})}$$

We note that $\overline{z}_1$ should be < 0 to satisfy the rigid body unilateral contact constraints in the limit. For the range of orientations $0 \leq \theta \leq \pi$ it is positive for $\tau = 0$ (no external forces or torques). Additionally, for the rolling contact case the steady state value of the tangential contact force must fall within the Coulomb bounds ($|\lambda_T| \leq \mu\lambda_N$). For the case presented this condition translates to:

$$|\frac{\lambda_T}{\lambda_N}| = |\frac{k_T\overline{z}_2}{k_N\overline{z}_1}| \leq \mu \quad (21)$$

Again using the time scale transformation $t = \sqrt{\epsilon}\tau$ and the coordinate transformation $y = z - \overline{z}$ gives the boundary layer dynamics:

$$y'' + BDy' + KDy = 0$$

with

$$K = \begin{bmatrix} k_N & 0 \\ 0 & k_T \end{bmatrix}, B = \begin{bmatrix} b_N & 0 \\ 0 & b_T \end{bmatrix}, D = \begin{bmatrix} \frac{1}{m} + \frac{l_1^2 C^2x}{I} & -\frac{l_1^2 CxSx}{I} \\ -\frac{l_1^2 CxSx}{I} & \frac{1}{m} + \frac{l_1^2 S^2x}{I} \end{bmatrix} \quad (22)$$

The eigenvalues of the matrix D are $\gamma_1 = \frac{1}{m}$, $\gamma_2 = \frac{1}{m} + \frac{l_1^2}{I}$ which are always positive real. Thus, the boundary layer is always stable.

4.3 Discussion

No contact This is a trivial case in which principles of classical rigid body dynamics show uniqueness and existence, and stability is not an issue.

Sliding contact Depending on the value of A and b, the LCP can have different outcomes as shown in Table 1. An examination of this table reveals that there is one case where two solutions will satisfy the LCP. This occurs when $A < 0$ and $b > 0$. The two solutions correspond to maintaining and breaking contact respectively. The no contact solution is obviously stable. If the case where the contact is maintained is examined via the singular perturbation analysis (note that $\overline{z} = A^{-1}b < 0$), since $A < 0$ this solution is unstable. In this situation, there is only one stable solution. However, this may not always be the case as illustrated next.

Rolling contact Consider the initial conditions and the parameters: $\dot{\theta} = 0\frac{rad.}{s}$, $m = 1kg$, $l_1 = 0.5m$, $l_2 = 0.5m$, $I = \frac{1}{12}m(l_1 + l_2)^2$ and $\tau = [-1\ -mg\ 4]^T\ N$. A plot illustrating the number of LCP solutions for the range of orientation angles $0 \leq \theta \leq \pi$ and coefficients of friction $0 \leq \mu \leq 2.5$ is given in Figure (3). We see that for a range of conditions the problem actually yields all 3 possible rigid body solutions

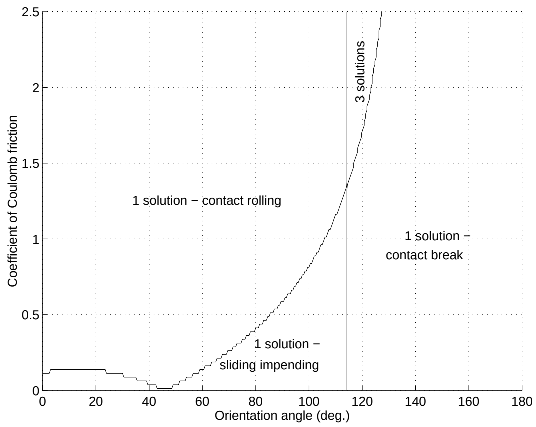

Figure 3: Planar rolling rod LCP solutions, $\frac{ml_1^2}{I} = 3$, $\dot{\theta} = 0$.

— contact break, rolling, and sliding. For instance at $\theta = 115°$, $\mu = 1.4$ the three solutions are: impending break ($\eta_N = 0.3328\frac{m}{s^2}$, $\eta_T = 20.7514\frac{m}{s^2}$), maintain rolling ($\lambda_N = 5.6729N$, $\lambda_T = -7.872N$), and impending sliding ($\lambda_N = 4.5673N$, $\lambda_T = -6.394N$, $\eta_T = 3.8489\frac{m}{s^2}$). The singular perturbation analysis shows that the contact breaking and rolling cases are both stable. The impending sliding case cannot be analyzed using this model because λ is not a differentiable function of the state.

5 Concluding remarks

It is well-known that there are difficulties in using rigid body dynamic models for the dynamic simulation of systems with frictional contacts. In particular, when rigid body models are used in conjunction with Coulomb's empirical law of friction for dynamic simulation of systems with frictional contacts, there may be situations in which there are no solutions or multiple solutions for the contact forces and the accelerations. In this paper, we describe a contact model that models the small compliance in the normal and tangential direction. We show that this compliant contact model, when used with the rigid body dynamic equations of motion, always yields a unique solution for the accelerations and the forces. While this model is superior to the traditional rigid body model in terms of accuracy and robustness, it is also more complex and requires a larger number of parameters. Therefore, it is appealing to use rigid body models, whenever concerns of uniqueness and existence do not arise. We use methods of singular perturbation theory to establish conditions under which solutions from the rigid body model are stable, or in other words, conditions in which the compliant contact model solution converges exponentially to the rigid body model solution. The basic ideas of this paper are applicable to any situation with frictional contacts. However, the rigid body model, and therefore the perturbation analysis, cannot be applied to statically indeterminate systems.

In situations when rigid body LCP analysis reveals multiple solutions, one might ask if stability analysis can help resolve the difficulty with ambiguities. We may simply discard the unstable solutions and retaining the stable one. However, as our example with rolling contact showed, there are also cases where there may be more than one stable solution or where there might be a unique, but unstable solution. Thus, the stability analysis simply shows when it is essential to pursue the more sophisticated compliant contact model, and when it is satisfactory to neglect the fast dynamics.

Our future work addresses incorporating stability analysis as a diagnostic tool in real-time simulation where it is prudent to check for stability and warn the user in unstable regimes.

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