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SIMPLIFIED DISPERSION RELATIONS FOR FLOQUET WAVES IN A PLATE WITH MULTIPLE ARRAYS OF ATTACHMENTS

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ABSTRACT

Recently, the authors developed analytical expressions for the dispersions of Floquet waves that propagate in a structure consisting of a plate with multiple arrays of line attachments. The dispersions of these Floquet waves, and in particular the imaginary parts of their wavenumbers, quantify the attenuation of vibrational energy in space as the frequency of a local excitation is varied. Understanding how the parameters of the attached structures, such as their spacings and impedances, affect the Floquet wave dispersions could provide further means to include consideration of energy localization or distribution in the structural design process. Such an understanding is developed in the present work by identifying those cases in which the treatment of certain arrays can be greatly simplified. In particular, limiting cases of small and large array spacings are investigated for which the treatment of particular arrays can be greatly simplified. Such simplifications are not immediately obvious without access to analytical expressions for the Floquet wavenumbers, as the dynamics of all arrays are coupled through the plate. Results presented here will aid the structural design community by indicating which design changes most effectively control energy distribution and by indicating when simplified finite element models of multiple-array structures are possible.

NOMENCLATURE

d_r Distance between adjacent substructures of the r^{th} array.
E Young's modulus of the plate material.
F_r Force applied by the plate to the r^{th} array.
k Transform wavenumber.
$k_{d,r}$ Spacing wavenumber of the r^{th} array (equation 8).
k_f Flexural wavelength of the bare plate.
m Mass per unit area of the bare plate.
Q Dispersion function (equation 12).
$\tilde{v}$ Wavenumber transform of plate velocity.
$\tilde{Y}$ Bare plate admittance in the wavenumber domain.
$\tilde{Y}_r$ Plate admittance in the wavenumber domain with the first $(r-1)$ arrays attached (equation 14).
Z_r Line impedance looking into the r^{th} array.
$Z^{(r)}$ Distributed impedance looking into the r^{th} array (equation 3).
ν Poisson's ratio of the plate material.
Ω Normalized frequency (equation 43).
Ω_{bound} Bounding frequency that indicates when an array may be homogenized to engineering accuracy.
ζ_r Normalized impedance (equation 27).

1 INTRODUCTION

A variety of engineering structures consist of a homogeneous elastic master structure, such as a plate or shell, that is attached to regularly spaced substructures, such as ribs, stringers, or fins. Due to other design criteria, such structures sometimes have two or more arrays of attachments, where each array consists of regularly spaced substructures that present identical impedances to the

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master structure. For example, aircraft fuselages are reinforced by ribs and stringers that prevent the structure from collapse and sections of the fuselage are joined by bulkheads. Ship structures are similarly constructed. In addition to these, a broad class of circular structures exist, such as brake rotors and gas turbines, that have one periodicity corresponding to the circumference of the structure and other periodicities that result from attached structures, such as heat fins or blades. This paper is concerned with the distribution of vibrational energy in such structures and, in particular, the effects of excitation frequency, master structure dynamics, substructure dynamics, and substructure spacing on the distribution of vibrational energy.

One way of understanding the distribution of vibrational energy is to look at the attenuation of waves as they propagate through the structure. The affects of periodic variations on the propagation of waves through a medium was observed by Brillouin (1946), who identified "stopping" and "passing" frequency bands where energy was localized and distributed, respectively. Mathematically, this phenomenon can be identified from Floquet's Theorem. Denoting $v(x)$ as the steady-state response amplitude of a time-harmonic response, this theorem states that $v(x) = p(x)\exp(ikx)$, where $v(x)$ is the complex amplitude of response, $p(x)$ is a periodic function whose period is the spatial period of the structure, and k is the Floquet wavenumber. When k has a large imaginary part, energy is localized because the Floquet wave is highly attenuated. Floquet wave dispersions were analytically found by Miles (1956) for the case of a beam on rigid supports and by Heckl (1961) for the case of a plate supported by regularly spaced beams. These analyses indicated that the imaginary part of k alternates between large and small as frequency is varied, thus producing stop and pass bands.

Following these works, two approaches emerged for analyzing more complicated structures. The first method, which shall be referred to here as the *eigenvalue method*, formulates and solves an eigenvalue problem for the attenuation constant based on Floquet's Theorem and equations of motion for one cell of the periodic structure. Applications of this method to beams with one array of attachments were described by Ungar (1966) and Bobrovnitskii and Maslov (1966). Mead (1970, 1973, 1975, 1996) extended the approach to any linear structure and developed insights into the locations of the stop and pass bands as well as the number of Floquet waves that propagate in a structure. The second method, which shall be referred to here as the *wavenumber method*, proceeds by taking the spatial Fourier transform of the differential equations of motion of the structure. Applications of the method to elastic structures were presented by Romanov (1971), Evseev (1973), and Rumerman (1975).

Both methods have been previously applied to the analysis of structures with two arrays of attachments. The eigenvalue method was employed by Gupta (1972) and the wavenumber analysis of a plate with two arrays of line attachments was first presented by Mace (1980). Recently, Cray (1994) presented an analysis of a plate with two arrays of attachments, one being arbitrarily shifted with respect to the other.

Recently, the authors developed a wavenumber-based procedure for finding Floquet wave dispersion relations for structures with multiple arrays of attachments, and applied it to the case of an elastic plate with multiple arrays of line attachments (Gueorguiev *et al*, 1999). The dispersion relation is unique in that it contains the dispersions as a sequence of structures, starting with a bare plate and conceptually adding the arrays in order of increasing spacing until one reaches the array with the largest spacing.

To further simplify the dispersion analysis of multiple array structures, this paper identifies conditions under which particular arrays can either be omitted or approximated insofar as the Floquet wave dispersions are concerned. For arrays with small spacings, the particular approximation studied here is that of homogenization, in which an array is approximated by evenly distributing the impedance of each attachment over a length equal to the spacing of the attachments. The distributed impedance is then combined with the plate impedance, so that the net effect of the array is to change the impedance of the plate. This approximation allows one to effectively replace the actual plate and attached array by a fictitious equivalent plate whose parameters are chosen so that it's dynamics approximate those of the actual structure. This approximation is sometimes referred to as a "smearing" of the array into the plate.

These simplifications in complexity are intended to aid the structural design community in two important respects. First, designers will be able to assess the importance of attachment spacing and impedance on the distribution of vibrational energy. In particular, this paper identifies those cases for which the spacing of the attachments is unimportant insofar as vibrational energy distribution is concerned. For these cases, designers could choose the spacings based on other design criteria. Second, the concepts will allow for simpler finite element models of periodic structures by indicating situations where a detailed model of a particular array is not necessary.

In the following section, the dispersion relations for a plate with multiple arrays of line attachments are reviewed. Section 3 presents the limiting case where the array spacings approach zero. In this limit, the array is homogenized into the plate and an expression is presented for the equivalent plate impedance. Section 4 establishes conditions under which the array may be homogenized into the plate, and bounds the associated errors, by expanding the Floquet wavenumbers in Taylor series about the bare plate's flexural wavenumber. The first term in the Taylor series is associated with the homogenization of the array into the plate and the higher-order terms represent the associated errors. Section 5 contains an investigation of the opposite limit, where the largest array spacing approaches infinity. For this case, it is shown that the array does not affect the Floquet wave dispersions and can therefore be entirely neglected from the model. Section 6 demonstrates through examples how these limiting cases are ac-

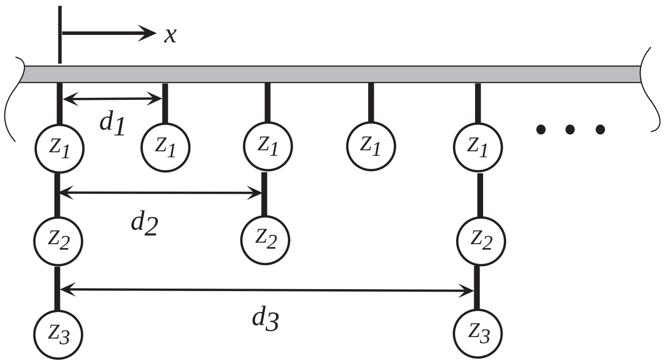

Figure 1. Thin elastic plate with three arrays of line attachments. The circles represent line impedances that extend along the y coordinate (into page).

tually applicable to the analysis of practical structures with finite array spacings.

2 EXACT DISPERSION RELATIONS FOR A PLATE WITH MULTIPLE ARRAYS

Figure 1 contains a schematic of the type of structure being considered here. It consists of a thin elastic plate with R arrays of line attachments. It is assumed that all attachments in the r^{th} array are identical and that each attachment exerts a line force that is proportional to its velocity and its impedance Z_r. The arrays are ordered by their spacings so that $d_r > d_{r-1}$. Furthermore, it is assumed that each array spacing is an integral multiple of the smaller array spacings and that a spatial point of coincidence exists at which an attachment from each array interacts with the plate. The dispersion relations of Floquet waves propagating in this structure have been previously derived in detail (Gueorguiev *et al*, 1999), so the derivation will only be summarized here.

It will be implicitly assumed that the plate's displacement is only a function of the x coordinate. The plate's displacement $W(x,t)$ is assumed to obey the following partial differential equation:

$$D\frac{\partial^4 W}{\partial x^4} - m\frac{\partial^2 W}{\partial t^2} = -\sum_{r=1}^{R} F_r(x,t), \tag{1}$$

where m is the mass per unit area of the plate. The bending stiffness is $D = Eh^3/[12(1-\nu^2)]$, where E is the Young's modulus, ν is the Poisson's ratio, and h is the plate thickness. Each force $F_r(x,t)$ represents the force applied by the plate to the r^{th} array and is given by

$$F_r(x,t) = Z^{(r)}(x)V(x,t) \tag{2}$$

where $V(x,t)$ is the plate velocity, related to the displacement by $V(x,t) = \partial W(x,t)/\partial t$. The distributed impedance of all attachments in the rth array is defined as

$$Z^{(r)}(x) = Z_r \sum_{n=-\infty}^{\infty} \delta(x - nd_r). \tag{3}$$

Writing the time-harmonic plate velocity as

$$V(x,t) = \mathrm{Re}\{v(x)\exp[-i\omega t)]\} \tag{4}$$

and substituting this into equation 1, one obtains an ordinary differential equation involving infinite sums over the attachment positions of all arrays. Material damping is included by allowing the Young's modulus to become complex valued, so that E is replaced by $E_0(1-i\eta)$, where η is the material loss factor. The array impedances are implicitly assumed to be functions of ω. One proceeds by taking the spatial Fourier transform, defined by

$$\tilde{v}(k) = \int_{-\infty}^{\infty} v(x)e^{-ikx}\,dx, \tag{5}$$

of the equation of motion. This transforms the summations over positions to summations over wavenumbers. Invoking Poisson's summation formula, equation 1 becomes

$$\tilde{v}(k) = -\tilde{Y}(k)\sum_{r=1}^{R} Z_r \tilde{v}_{\Sigma,r}(k) \tag{6}$$

where the summed plate velocities are

$$\tilde{v}_{\Sigma,r}(k) = \frac{1}{d_r}\sum_{n=-\infty}^{\infty} \tilde{v}(k - nk_{d,r}), \tag{7}$$

the spacing wavenumber is

$$k_{d,r} = \frac{2\pi}{d_r}, \tag{8}$$

the wavenumber admittance of the plate is

$$\tilde{Y}(k) = \frac{1}{-im\omega}\left(\frac{k_f^4}{k_f^4 - k^4}\right), \tag{9}$$

and the flexural wavelength is

$$k_f = \left(\frac{m\omega^2}{D}\right)^{1/4} \tag{10}$$

After many algebraic manipulations, equation 6 is cast into the form

$$Q(k)\tilde{v}(k) = 0, \tag{11}$$

where the dispersion function $Q(k)$ is given as

$$Q(k) = \prod_{r=1}^{R}[1 + Z_r\tilde{Y}_{\Sigma,r}(k)] \tag{12}$$

and the summed admittances $\tilde{Y}_{\Sigma,r}$ are defined by

$$\tilde{Y}_{\Sigma,r}(k) = \frac{1}{d_r}\sum_{n=-\infty}^{\infty}\tilde{Y}_r(k - nk_{d,r}). \tag{13}$$

Each $\tilde{Y}_r$ represents the wavenumber admittance of the plate with the first $(r-1)$ arrays attached. The rth admittance satisfies the recursion relation,

$$\tilde{Y}_r(k) = \begin{cases} \tilde{Y}(k) & \text{for } r = 1 \\ \dfrac{\tilde{Y}_{r-1}(k)}{1 + Z_{r-1}\tilde{Y}_{\Sigma,r-1}(k)} & \text{for } r = 2,\ldots,R. \end{cases} \tag{14}$$

Requiring a nontrivial solution for $\tilde{v}(k)$ leads to the dispersion relation

$$Q(k) = 0. \tag{15}$$

The roots of the latter that lie in the strip defined by $0 < \Re(k) < 2\pi/d_r$ and $\Im(k) > 0$ are the Floquet wavenumbers of the plate with R arrays. Values of k that satisfy this equation are the Floquet wavenumbers. The particular form of $Q(k)$ given in equation 12 has an interesting interpretation. The rth term in the product represents the dispersion relation of the plate with the first r arrays attached.

Gueorguiev *et al* (1999) have shown that the dispersion relation for the plate with the first r arrays attached transforms into a quadratic equation given by

$$\cos^2(kd_r) + a_r\cos(kd_r) + b_r = 0, \tag{16}$$

where a_r and b_r are given in Appendix A. These constants depend on the Floquet wavenumbers of the plate with only the first $(r-1)$ arrays attached. Note that the Floquet wavenumbers of the bare plate are simply the plate's flexural wavenumbers, k_f and ik_f. One computes the wavenumbers of a plate with R arrays by sequentially evaluating equation 16 for each array, beginning with $r = 1$ and ending with $r = R$.

A parenthetical superscript on the Floquet wavenumbers shall be used to denote the number of attached arrays and a subscript of either 1 or 2 shall refer to one of the two types of Floquet waves. The first type has a small imaginary part in the pass bands of the structure, and is hence referred to as the *propagating Floquet wave*. It is analogous to the flexural wave on a bare plate. The second type has a large imaginary part at any frequency and is analogous to the evanescent wave on a bare plate, which is confined near concentrated loadings or discontinuities.

3 DISPERSION RELATIONS FOR ARRAYS WITH SMALL SPACINGS

In this section, an expression is derived for the wavenumber impedance of an equivalent plate whose dynamics approximate those of a plate with arrays of closely spaced attachments. For clarity, consider a structure with R arrays and first take the limit as the smallest array spacing, d_1, approaches zero. The relative impedance, defined by

$$Z_r^{rel} = \frac{Z_r}{d_r}, \tag{17}$$

shall be held constant in taking the limit of small array spacing. Physically, this amounts to fixing the total impedance attached to a finite length structure so that the impedance of each attachment must decrease if the spacing is decreased.

It shall be expedient to return to the equation of motion in the spatial domain. From equation 3, the impedance of the first ($r = 1$) array is written as

$$Z^{(1)}(x) = Z_1^{rel}d_1\sum_{n=-\infty}^{\infty}\delta(x - nd_1). \tag{18}$$

Fixing Z_1^{rel} and taking the limit as $d_1 \to 0$ gives

$$\lim_{d_1\to 0} Z^{(1)} = Z_1^{rel}\int_{-\infty}^{\infty}\delta(x-\eta)d\eta = Z_1^{rel} \tag{19}$$

The collective impedance $Z^{(1)}(x)$ approaches a constant when the corresponding spacing d_1 approaches 0.

After substituting equation 19 into equation 1 and taking the Fourier transform, the equation of motion becomes

$$\tilde{Z}(k)\tilde{v}(k) = -Z_1^{rel}\tilde{v}(k) - \sum_{r=2}^{R} Z_q \sum_{m=-\infty}^{\infty} \tilde{v}(k - mk_{d,r}), \quad (20)$$

where $\tilde{Z}(k) = \tilde{Y}^{-1}(k)$ defines the plate's wavenumber impedance. Equation equation 20 can be cast in the simpler form

$$\tilde{Z}_{eq}(k)\tilde{v}(k) = -\sum_{r=2}^{R} Z_r \sum_{m=-\infty}^{\infty} \tilde{v}(k - mk_{d,r}), \quad (21)$$

in which

$$\tilde{Z}_{eq}(k) = \tilde{Z}(k) + Z_1^{rel} \quad (22)$$

Upon analogously defining the equivalent wavenumber admittance as $Y_{eq}(k) = Z_{eq}^{-1}(k)$, the governing wavenumber equation becomes

$$\tilde{v}(k) = -\tilde{Y}_{eq}(k) \sum_{r=2}^{R} Z_r \tilde{v}_{\Sigma,r}(k), \quad (23)$$

which is identical to equation 6 except that they $r = 1$ array is removed and the plate admittance $\tilde{Y}(k)$ is replaced by the equivalent admittance $\tilde{Y}_{eq}(k)$. In summary, the R-array problem is transformed into an $(R-1)$-array problem where the $r > 1$ arrays are attached to a fictitious plate with an equivalent admittance $Y_{eq}(k)$. The $r = 1$ array has been effectively homogenized into the plate.

If we require all array spacings to approach zero, then the dispersion relation becomes

$$\tilde{Z}(k) + \sum_{r=1}^{R} {Z_r}^{rel} = 0 \quad (24)$$

The solutions of the latter represent the Floquet wavenumbers of the homogenized array, given by

$$k_1^{(R)} = k_f \left(1 + \sum_{r=1}^{R} \zeta_r\right)^{1/4} \quad (25)$$

and

$$k_2^{(R)} = ik_f \left(1 + \sum_{r=1}^{R} \zeta_r\right)^{1/4}, \quad (26)$$

where a normalized impedance has been defined as

$$\zeta_r = \frac{Z_r}{-i\omega m d_r}. \quad (27)$$

Physically, the denominator of ζ represents the impedance due to the mass of a section of plate whose length is equal to the spacing between attachments. If each attachment is modeled as a mass then the normalized impedances are positive real numbers.

4 CONDITIONS FOR HOMOGENIZING ARRAYS WITH SMALL SPACINGS

Physically, one expects that an array could be homogenized into the plate when the array spacing is small compared to a characteristic response scale of the plate. For example, homogenization would be reasonable if the bare plate's flexural wavelength, λ_f, is much greater than the array spacing, $\lambda_f \gg d_1$. One might also expect that the homogenization condition would involve the impedances of the attachments. For example, arrays with small impedances would be more accurately homogenized than large impedance arrays for a fixed frequency.

These expectations may be mathematically qualified by deriving approximations to the Floquet wavenumbers for small values of $k_f d_1$. In this section, this investigation is conducted by solving equation 16 for the Floquet wavenumbers and expanding them in a Taylor series about $k_f d_1 = 0$. Due to the complexity of the expressions for the Floquet wavenumbers, this analysis was verified by a symbolic manipulation software. First, let us consider a simple structure consisting of a plate with one array of attachments ($R = 1$). The Taylor series expansions of the two Floquet wavenumbers are

$$k_1^{(1)} \sim k_f (1+\zeta_1)^{1/4} + \frac{(\zeta_1)^2}{2880 d_1 (1+\zeta_1)^{3/4}} (k_f d_1)^5 + \frac{{\zeta_1}^2}{12096 d_1 (1+\zeta_1)^{1/4}} (k_f d_1)^7 + \mathcal{O}[(k_f d_1)^9] \quad (28)$$

and

$$k_2^{(1)} \sim ik_f (1+\zeta_1)^{1/4} + \frac{i{\zeta_1}^2}{2880 d_1 (1+\zeta_1)^{3/4}} (k_f d_1)^5 - \frac{i{\zeta_1}^2}{12096 d_1 (1+\zeta_1)^{1/4}} (k_f d_1)^7 + \mathcal{O}[(k_f d_1)^9] \quad (29)$$

One immediately recognized the leading terms of these expansions as the homogenized wavenumbers given in equations 25 and 26 with $R = 1$. The remaining higher-order terms represent errors in the homogenization. Recognizing that the second term contributes most to the error, the following homogenization condition is arrived at:

$$|k_f d_1| \ll 7.33 \left| \frac{(1+\zeta_1)^{1/4}}{\sqrt{\zeta_1}} \right| \tag{30}$$

Since the right hand side of Equation 30 is decreasing with increasing ζ_1, it follows that increasing the relative impedance reduces the frequency range in which the array can be homogenized. In fact, in the limit as $\zeta_1 \to \infty$, no homogenization is possible. This makes physical sense as the limit corresponds to pinning the plate at the attachment points. As an engineering measure of the accuracy of the homogenization, equation 30 may be cast in the form

$$|k_f d_1| \leq 7.33 C_1 \left| \frac{(1+\zeta_1)^{1/4}}{\sqrt{\zeta_1}} \right|. \tag{31}$$

For our example involving masses attached to lightly damped plates, choosing $C_1 = 0.15$ gave errors in the Floquet wavenumbers of less than one percent.

Similar results were obtained for a plate with two arrays of attachments ($R = 2$), where both arrays are homogenized. The Floquet wavenumbers for this case are

$$k_1^{(2)} \sim k_f (1+\zeta_1+\zeta_2)^{1/4} + \frac{{\zeta_1}^2 + 2\zeta_1\zeta_2 + {n_2}^4{\zeta_2}^2}{2880 d_1 (1+\zeta_1+\zeta_2)^{3/4}} (k_f d_1)^5 + \frac{{\zeta_1}^2 + 2\zeta_1\zeta_2 + {n_2}^6{\zeta_2}^2}{12096 d_1 (1+\zeta_1+\zeta_2)^{1/4}} (k_f d_1)^7 + \mathcal{O}[(k_f d_1)^9] \tag{32}$$

and

$$k_2^{(2)} \sim i k_f (1+\zeta_1+\zeta_2)^{1/4} - i\frac{{\zeta_1}^2 + 2\zeta_1\zeta_2 + {n_2}^4{\zeta_2}^2}{2880 d_1 (1+\zeta_1+\zeta_2)^{3/4}} (k_f d_1)^5 - i\frac{{\zeta_1}^2 + 2\zeta_1\zeta_2 + {n_2}^6{\zeta_2}^2}{12096 d_1 (1+\zeta_1+\zeta_2)^{1/4}} (k_f d_1)^7 + \mathcal{O}[(k_f d_1)^9] \tag{33}$$

where n_2 denotes the spacing ratio d_2/d_1.

The leading terms of these expansions will dominate the second terms if

$$|k_f d_1| \ll 7.33 \left| \frac{(1+\zeta_1+\zeta_2)^{1/4}}{\left({\zeta_1}^2 + 2\zeta_1\zeta_2 + {n_2}^4{\zeta_2}^2\right)^{1/4}} \right| \tag{34}$$

One observes from this condition that an increase of the spacing ratio n_2 or the relative impedances $\zeta_{1,2}$ would reduce the frequency range over which homogenization is possible. As before, equation 34 may be cast in the form

$$|k_f d_1| \leq 7.33 C_2 \left| \frac{(1+\zeta_1+\zeta_2)^{1/4}}{\left({\zeta_1}^2 + 2\zeta_1\zeta_2 + {n_2}^4{\zeta_2}^2\right)^{1/4}} \right|. \tag{35}$$

For our example involving masses attached to lightly damped plates, choosing $C_2 = 0.12$ gave errors in the Floquet wavenumbers of less than one percent.

5 DISPERSION RELATIONS FOR ARRAYS WITH LARGE SPACINGS

In this section, the limit of large array spacing is examined. The analysis presented here assumes that d_R, which is the largest array spacing, is larger than any other spatial scale in the problem. For this case, one finds that the R^{th} array may be entirely neglected in analyzing the wave dispersions. The physical reason for this is that the Floquet waves attenuate and are therefore not able to propagate the large distances between attachments in the R^{th} array. At the end of this section, the results will be generalized to the case where the R and $R-1$ arrays have large spacings.

First, we define a multiple of the R^{th} array's spacing wavenumber as

$$\xi_n = n k_{d,R} = n\frac{2\pi}{d_R} \tag{36}$$

and rewrite the spacing wavenumber as a difference between multiples,

$$\Delta\xi = \xi_{n+1} - \xi_n = k_{d,R}. \tag{37}$$

Substituting these into equation 13 yields

$$\tilde{Y}_{\Sigma,R}(k) = \frac{\Delta\xi}{2\pi} \sum_{n=-\infty}^{\infty} \tilde{Y}_R(k-\xi_n) \tag{38}$$

Taking the limit as $d_R \to \infty$ gives

$$\lim_{d_R\to\infty} \tilde{Y}_{\Sigma,R}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{Y}_R(\xi)d\xi = Y_R. \tag{39}$$

As shown, this integral may be interpreted as an inverse Fourier transform of $\tilde{Y}_R$ evaluated at $x = 0$. Specifically, Y_R represents the drive-point spatial impedance of the plate at $x = 0$ with the first $(R-1)$ arrays attached. Introducing this limit in equation 12 results in

$$Q(k) = (1+Z_R Y_R) \prod_{r=1}^{R-1} [1+Z_r \tilde{Y}_{\Sigma,r}(k)] \tag{40}$$

Since the term preceding the product is independent of wavenumber, the dispersion relation reduces to that of the plate with the first $(R-1)$ arrays attached.

Analogously, allowing both d_{R-1} and d_R to approach infinity, one can show that the dispersion relation function becomes

$$Q(k) = (1+Z_R Y_R)(1+Z_{R-1} Y_{R-1}) \prod_{r=1}^{R-2} [1+Z_r \tilde{Y}_{\Sigma,r}(k)] \tag{41}$$

where Y_{R-1} is the drive-point mobility of the plate with the first $R-2$ arrays attached. This result extends in the obvious way to any number of arrays whose spacings approach infinity.

6 NUMERICAL EXAMPLES

In this section, the limiting cases presented in the previous sections will be illustrated for the case of an elastic beam with two and three arrays of line attachments. The substructures in all arrays will be taken as line masses with the mass per unit length equal to the mass in a unit square of plate, so that

$$Z_r = -i\omega mh. \tag{42}$$

The wavenumber of the propagating Floquet wave will be plotted versus a normalized frequency defined by

$$\Omega = \omega d_R^2 \sqrt{\rho h/D}. \tag{43}$$

The plate parameters corresponding to steel are $E = 2\times 10^{11}$ Pa, $\rho = 7800$ kg/m^3, $\nu = 0.3$, and $\eta = 0.01$. The plate thickness is $h = 1$ cm. The width of the plate in the y coordinate is assumed small enough to allow the plate to be treated as a beam, which is accomplished by artificially setting the Poisson's ratio to zero in the bending rigidity, D.

The first two examples will illustrate the approximations and associated errors involved in truncating the Taylor series expansions of Floquet wavenumbers given in Section 4. In the first example, we consider a plate with a single array having spacing $d_1 = 1$. The real and imaginary parts of the propagating Floquet wavenumbers are shown in Figures 2 and 3. Also shown in these figures is the homogenization given by the leading term in equation 28. We find that the condition given in equation 31 is satisfied

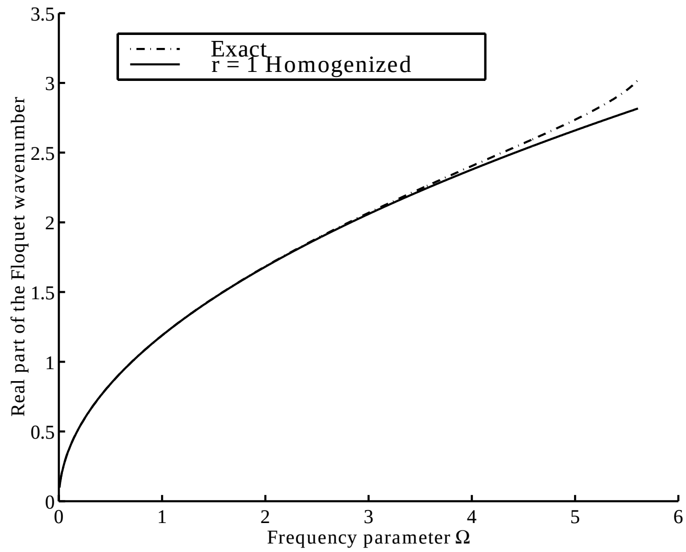

Figure 2. Real part of the propagating Floquet wavenumber for a structure with one array having a spacing $d_1 = 1$m.

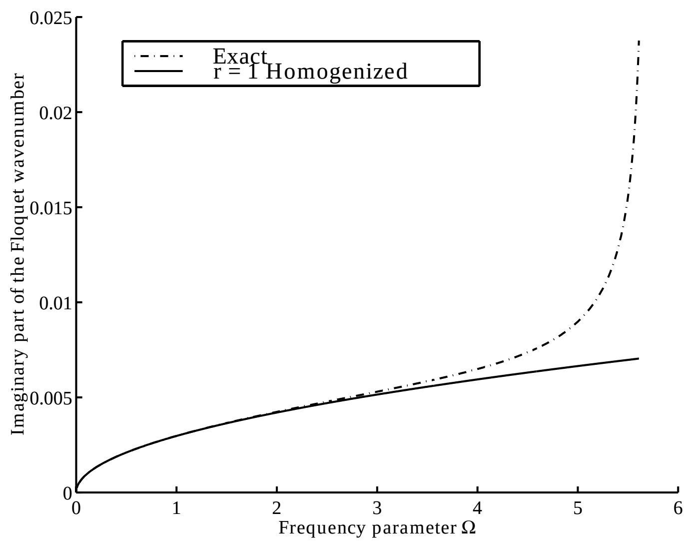

Figure 3. Imaginary part of the propagating Floquet wavenumber for a structure with one array having a spacing $d_1 = 1$m.

when $\Omega < \Omega_{bound}$, where $\Omega_{bound} = 2.9$ for this example. Below this bounding frequency, the wavenumber is well-approximated by the homogenized result.

The second example involves a structure with two arrays having spacings $d_1 = 1$ and $d_2 = 2$. The real and imaginary parts of the propagating Floquet wavenumber are given in Figures 4 and 5. Also shown is the homogenization obtained by only retaining the leading-order term in equation 33. Equation 35 is satisfied when

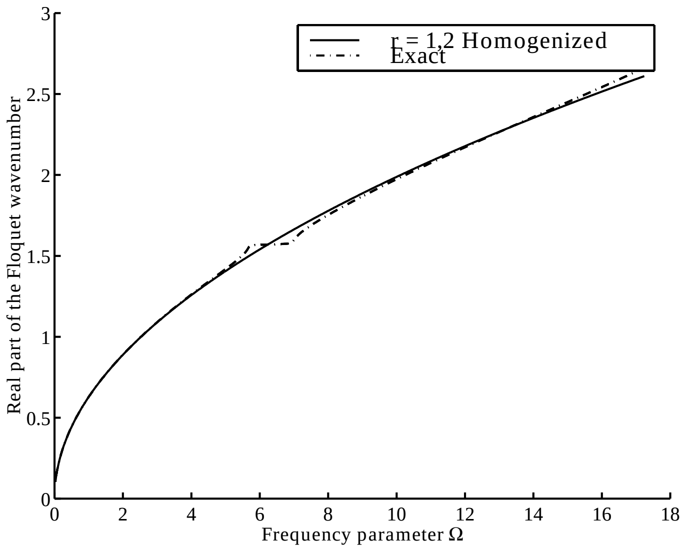

Figure 4. Real part of the propagating Floquet wavenumber for a structure with two arrays having spacings $d_1 = 1$m and $d_2 = 2$m.

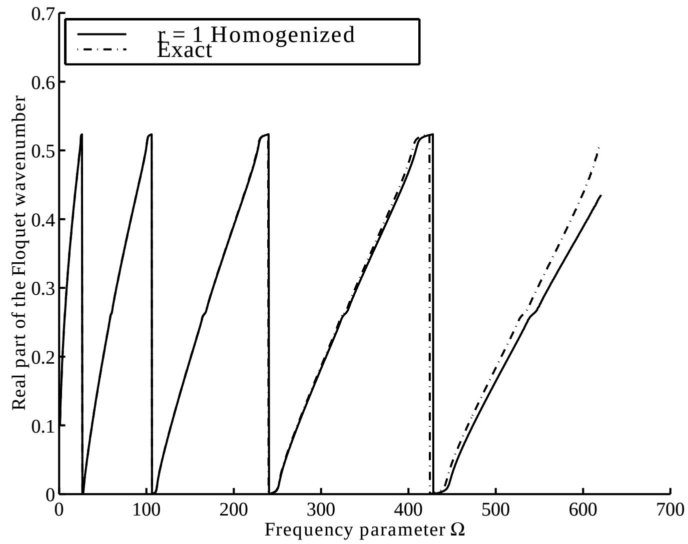

Figure 6. Real part of the propagating Floquet wavenumber for a structure with three arrays having spacings $d_1 = 1$m, $d_2 = 6$m, and $d_3 = 12$m.

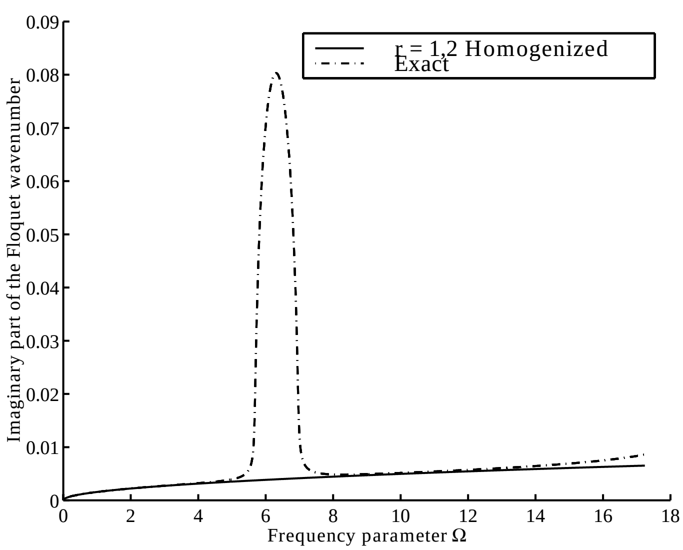

Figure 5. Imaginary part of the propagating Floquet wavenumber for a structure with two arrays having spacings$d_1 = 1$m and $d_2 = 2$m.

$\Omega < \Omega_{bound}$, where $\Omega_{bound} = 4$. Again, excellent agreement is found below this bounding frequency.

A more complex situation arises if one wishes to homogenize a subset of the arrays. In equation 23, it was shown that the first array ($r = 1$) could be homogenized into the plate while the other arrays ($r \geq 2$) were treated exactly. Unfortunately, simple expressions for the associated errors are difficult to obtain as one must track the approximations introduced by homogenizing the first array through the equations describing the other arrays. Nonetheless, we present an example here to illustrate that such an approach has a reasonable frequency range of applicability. Three arrays are attached to a plate with spacings $d_1 = 1$m, $d_2 = 6$m, and $d_3 = 12$m. The real and imaginary parts of the propagating Floquet wavenumber, computed by equation 16, are plotted in Figures 6 and 7. Also plotted is an approximation in which the first array ($r = 1$) has been homogenized into the plate according to equation 23. As in the previous examples, good agreement is obtained at low frequencies.

The large array-spacing limit is illustrated in Figure 8. In this figure, the imaginary part of the Floquet wavenumber is plotted for a plate with two arrays having spacings $d_1 = 1$ and $d_2 = 6$. The plot consists of a sequence of small stop bands at low frequency and a broad stop band centered at $\Omega \sim 275$. The small stop bands are caused by the large-spacing array ($r = 2$) and the large stop band is caused by the small-spacing array ($r = 1$). Taking the limit of large spacing would lead one to omit the large-spacing array ($r = 2$) from consideration. This result is labeled as the "one array" curve in the plot and agrees with the "two array" result at higher frequencies.

7 CONCLUSIONS

In this paper, two limiting cases of array spacing have been examined. In the case of small array spacing, an expression was derived that allows one to homogenize the array into the plate. For large array spacings, it was demonstrated that the array may be neglected in the analysis of wave dispersions. Although these results were obtained as limiting cases, it has been shown through

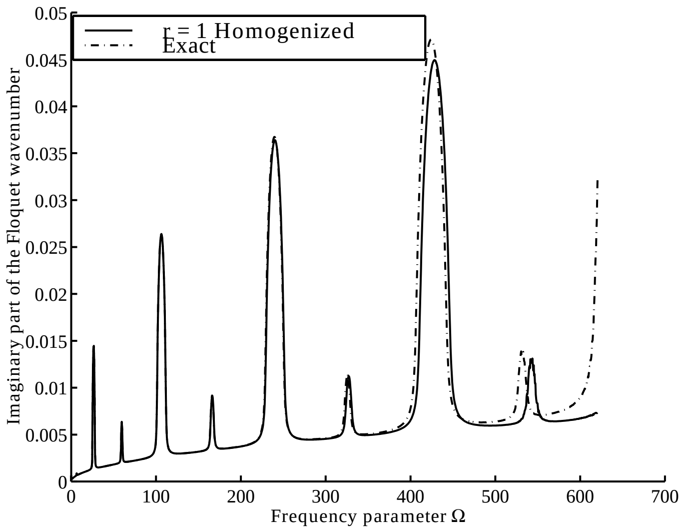

Figure 7. Imaginary part of the propagating Floquet wavenumber for a structure with three arrays having spacings $d_1 = 1$m, $d_2 = 6$m, and $d_3 = 12$m.

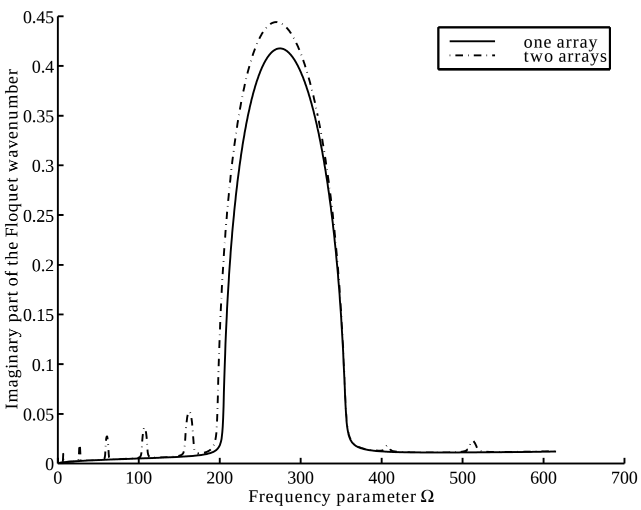

Figure 8. Imaginary part of the propagating Floquet wave in a plate with one ($d_2 = 6$) and two arrays of line attachments ($d_1 = 1$ and $d_2 = 6$). The one-array case corresponds to the large spacing limit.

examples that these cases apply to practical structures with finite array spacings.

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A CONSTANTS IN THE DISPERSION RELATION

The constants a_r and b_r appearing in equation 16 are related to the Floquet wavenumbers of the plate with the first $r-1$ arrays,

$$\begin{aligned} a_r &= -\cos(k_1{}^{(r-1)}) - \cos(k_2{}^{(r-1)} d_r) \\ &\quad - Z_r A_r \sin(k_1{}^{(r-1)} d_r) - Z_r B_r \sin(k_2{}^{(r-1)} d_r) \end{aligned} \tag{44}$$

and

$$\begin{aligned} b_r &= \cos(k_1^{(r-1)} d_r)\cos(k_2^{(r-1)} d_r) \\ &\quad + Z_r A_r \sin(k_1^{(r-1)} d_r)\cos(k_2^{(r-1)} d_r) \\ &\quad + Z_r B_r \sin(k_2^{(r-1)} d_r)\cos(k_1^{(r-1)} d_r) \end{aligned} \tag{45}$$

where A_r and B_r are also related to the Floquet wavenumbers of the plate with first $r-1$ arrays and are given as

$$A_r = \begin{cases} \dfrac{k_f}{4im\omega} & \text{if } r = 1 \\ \dfrac{\tilde{Y}_{\Sigma,r-1}(k_1^{(r-1)})}{Z_{r-1}^{rel}\tilde{Y}'_{\Sigma,r-1}(k_1^{(r-1)})} & \text{if } r = 2,\ldots,R \end{cases} \tag{46}$$

and

$$B_p = \begin{cases} \dfrac{k_f}{4m\omega} & \text{if } r = 1 \\ \dfrac{\tilde{Y}_{\Sigma,r-1}(k_2^{(r-1)})}{Z_{r-1}^{rel}\tilde{Y}'_{\Sigma,r-1}(k_2^{(r-1)})} & \text{if } r = 2,\ldots,R \end{cases}. \tag{47}$$

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