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Model Reduction Techniques for Shock Loaded Equipment Emulators

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In a variety of situations, an undesired shock excitation is applied to a master structure that supports shock-sensitive equipment. Often, one wishes to design and test a master structure that transmits the least amount of shock energy to the attached equipment. In scaled testing of new designs, a major task is to design and construct "equipment emulators" – inexpensive mechanical systems which approximately mimic the dynamic behavior of the actual full-scale equipment as seen by the master structure. The method of balanced truncation is presented here as means by which reduced-order equipment emulators with specified error bounds can be designed. The proposed approach uses easily obtainable frequency-domain impedance descriptions of the master structure and actual equipment at the attachment points. The method is illustrated through application to two simple examples.

INTRODUCTION

In recent years, there has been a shift in emphasis from the design of equipment that can withstand high shock loads to the design of structures on which commercial, off the shelf (COTS) equipment can survive. Given the prohibitive cost of testing structural design concepts using full-scale prototypes, finite element models and scaled mechanical models are often employed. In either case, the equipment is typically the most dynamically complex component of the system. Consequently, the construction of detailed numerical or scaled mechanical equipment models can be both difficult and costly. As an alternative, reduced-order equipment models can provide a means of including salient dynamics in the overall model at modest cost. The modeling effort is aided by the fact that the equipment is usually joined to the supporting structure at a small number of attachment points. Thus, the reduced model need only reproduce the input-output behavior at these locations.

The requirements of such a model reduction scheme are as follows. It must be possible to quantitatively characterize the tradeoff between model complexity and modeling error. In particular, modeling error should be expressed by a metric appropriate to the testing goals. For shock qualification, the appropriate metric is the shock spectrum computed from the motion of the master structure at the equipment attachment points. The technique should be based on an "actual" model easily obtained by experiment. Furthermore, the reduced model must be amenable to implementation. For scaled mechanical tests, this means that it must be possible to fabricate the reduced model.

Prior work on the design of scaled mechanical equipment emulators is limited to acoustic performance. Two approaches have been employed in the modeling of equipment cabinets: modal reduction and exact miniaturization. The former consisted of reproducing the first four fixed-base modal frequencies and masses. Design refinement involved adding damping materials to the nominal design to minimize the difference between drive-point impedance of the actual and scaled equipment at the attachment points (considered individually) [1]. The latter method involved the fabrication of a scaled cabinet with a variety of oscillators attached to the shelves [1]. With regard to shock, Barbone and co-workers have developed numerical equipment models, applicable to modally dense systems, which are described by a small number of physically motivated parameters [2,3]. These models have quantifiable error bounds and accurately reproduce early-time relations between forces and displacements at the attachment points. At this time, these techniques have not been evaluated with respect to shock spectrum error, however.

As an alternative to the approaches described above, model reduction can be cast as an optimization problem. Intermediate to solving an optimization problem, balanced truncation provides a simple technique for truncating model states based on their contribution to output energy. For equipment emulation, the output vector is composed of attachment point velocities. The method addresses shock spectrum error indirectly by providing a bound on velocity error energy. The technique and its error bound are described in the following section. Balanced reduction is then applied to two example systems in the subsequent section. Its performance is compared with modal reduction in the time domain as well as in the context of shock spectra. Conclusions are presented in the final section of the paper.

BALANCED TRUNCATION

As in [1,2,3], it is assumed that the input-output behavior of the equipment at its attachment points can be represented using a linear model. If the equipment is attached to the structure by a shock or vibration mount, this implies the use of an above-mount equipment model. The use of linear models is also motivated by the observation that scaled testing is often conducted using nondestructive input amplitudes to allow for repeated trials. With this assumption, the equipment can be expressed as a first-order state space system given by

$$\dot{x}(t) = Ax(t) + bu(t), \quad y(t) = Cx(t) \tag{1}$$

where $u \in R^M$ is a vector of inputs (attachment point forces), and $y \in R^M$ is a vector of outputs (attachment point velocities). The vector $x \in R^N$ is referred to as the state vector, which consists of displacement and velocity quantities for a discretized model of the emulator. The number of attachment points is far fewer than the number of degrees of freedom of the model, so that $M << N$. The input-output transfer function $y(s)=G(s)u(s)$ associated with (1) is

$$G(s) = C(sI - A)^{-1}B \tag{2}$$

which can be obtained experimentally from measurements of drive-point and transfer admittance. The triplet of matrices (A, B, C) is called a realization of $G(s)$. While the transfer function is unique, the realization is not. If T is an invertible matrix, then the triple (TAT^{-1}, TB, CT^{-1}) is another realization. The goal of model reduction is to solve for an approximate model $\hat{G}(s)$ possessing $L < N$ system states while introducing the least error in the transfer function $G(s)$. Error can be described by $\left\| G(s) - \hat{G}(s) \right\|$ where $\|\cdot\|$ is an induced matrix norm.

In the field of structural dynamics, modal realizations are often employed in model reduction. For a modal realization, TAT^{-1} is block diagonal, so that

$$TAT^{-1} = \mathrm{diag}\left(\begin{bmatrix} -\zeta_1\omega_1 & -\sqrt{1-\zeta_1^2}\,\omega_1 \\ \sqrt{1-\zeta_1^2}\,\omega_1 & -\zeta_1\omega_1 \end{bmatrix}, \ldots, \begin{bmatrix} -\zeta_{N/2}\omega_{N/2} & -\sqrt{1-\zeta_{N/2}^2}\,\omega_{N/2} \\ \sqrt{1-\zeta_{N/2}^2}\,\omega_{N/2} & -\zeta_{N/2}\omega_{N/2} \end{bmatrix} \right) \tag{3}$$

Each 2×2 submatrix corresponds to a pair of the N system states. In modal truncation, pairs of states are eliminated from the model based on a criteria such as rate of decay, $\zeta_i\omega_i$, frequency, $\sqrt{1-\zeta_i^2}\,\omega_i$, or transfer function error. The latter quantity is given by $\|C_i B_i\| / |\zeta_i\omega_i|$ [6]. Transfer function error is the most appropriate of these for equipment model reduction. Modal truncation has the advantage of being conceptually simple. In addition, it provides a physical interpretation of the system states as modes. The technique can become impractical, however, when a structure possesses many modes whose contributions to the transfer function error are comparable. This is true of modally dense systems, such as COTS equipment cabinets [1].

Given the goal of model reduction, truncation should ideally be performed on the realization in which the retained states minimize the error norm, $\left\| G(s) - \hat{G}(s) \right\|$. Clearly, this optimal choice of realization will depend both on $G(s)$ and on the number of states to be retained, L. Furthermore, there is no reason to assume that a modal realization is close to the optimal. In contrast, a balanced realization is one in which the states are selected according to input-output energy transfer. Those states least involved in energy transfer are truncated. While not optimal, it is shown below that this method provides excellent approximation of modally dense structures.

A description of a balanced realization for the transfer function $G(s)$ requires the introduction of the controllability gramian P and observability gramian Q. These gramians satisfy the equations:

$$\begin{aligned} AP + PA^T + BB^T &= 0 \\ A^T Q + QA + C^T C &= 0 \end{aligned} \tag{4}$$

The observability gramian has the interpretation (see, e.g., [4]) offered by the following calculation. Given the input $u(t) = 0,\ t > 0$ and initial state $x(0) = x_0$, then $\int_0^\infty y(t)^T y(t) = x_0^T Q x_0$. If Q has certain very small eigenvalues, then the initial conditions corresponding to those eigenvectors will have very little effect upon the output. The controllability gramian can be interpreted by calculating the minimum control energy which was needed to move the state vector x from the origin to its initial value $x(0) = x_0$. Mathematically, this can be posed as the following optimization problem. Find $J(u_{opt}) = \min_{u \in \mathcal{L}_2(-\infty,0)} \int_{-\infty}^0 u^T(t) u(t) dt$ subject to (1) and $x(0) = x_0$. The solution is given by $J(u_{opt}) = x_0^T P^{-1} x_0$ [6]. Thus, if certain eigenvalues of P are very small, then the states $x(0)$ associated with those eigenvalues are very difficult (control costly) to achieve. Assuming $u(t) = 0,\ t > 0$, these results can be combined to yield the ratio of future output energy to prior minimum input energy associated with an arbitrary initial state, x_0 [6].

$$\sup_u \frac{\int_0^\infty y(t)^T y(t)}{\int_0^\infty u(t)^T u(t)} = \frac{x_0^T Q x_0}{x_0^T P^{-1} x_0} = \frac{w^T P^{1/2} Q P^{1/2} w}{w^T w},\ \ x_0 = P^{1/2} w \tag{5}$$

The eigenvalues of $P^{1/2} Q P^{1/2}$ can be seen to provide a means of ranking the importance of state space directions (described by eigenvectors) in terms of their 2-norm contribution to this energy. Consequently, a realization that diagonalizes $P^{1/2} Q P^{1/2}$ makes it possible to apply this ranking directly to the realization's states.

Such a realization $(\overline{A}, \overline{B}, \overline{C})$ is termed balanced and can be shown to satisfy

$$\begin{aligned} \overline{A}\Sigma + \Sigma \overline{A}^T + \overline{B}\overline{B}^T &= 0 \\ \overline{A}^T \Sigma + \Sigma \overline{A} + \overline{C}^T \overline{C} &= 0 \end{aligned} \tag{6}$$

The observability gramian and controllability gramians in this case are identical, diagonal matrices $\Sigma = \text{diag}(\sigma_1, \sigma_2, ...\sigma_N),\ \sigma_i > 0$. The realization is termed ordered if $\sigma_1 > \sigma_2 > .\sigma_N$.

Any asymptotically stable, minimal realization (A, B, C) with observability gramian Q and controllability gramian P can be transformed into a balanced realization (TAT^{-1}, TB, CT^{-1}) where $P = RR^T$ is a Cholesky factorization of P, $RQR^T = U\Sigma^2 U^T$ is a singular value decomposition of RQR^T, and $T = \Sigma^{1/2} U^T R^{-1}$ [6]. The truncation of a balanced realization results in a stable, minimal system. Let $(\overline{A}, \overline{B}, \overline{C})$ be an ordered, balanced realization with transfer function $G(s)$. The diagonal gramian Σ is partitioned as $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$, where $\Sigma_2 = \text{diag}(\sigma_{L+1}, \sigma_2, ...\sigma_N)$ consists of "small" elements. The matrices $(\overline{A}, \overline{B}, \overline{C})$ are partitioned conformably, so that

$$\overline{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\quad \overline{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\quad \overline{C} = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \tag{7}$$

The truncated system (A_{11}, B_1, C_1) is stable, with transfer function $\hat{G}(s)$. The approximation satisfies the bound [6]

$$\left\| G(s) - \hat{G}(s) \right\|_\infty \leq 2 \sum_{i=L+1}^{N} \sigma_i \tag{8}$$

where the infinity norm for the system $y(s) = G(s)u(s)$ is defined by

$$\left\| G(s) \right\|_\infty = \sup_u \frac{\int_0^\infty y(t)^T y(t)}{\int_0^\infty u(t)^T u(t)} \tag{9}$$

One technical difficulty with the technique described above is that if $G(s)$ is a passive transfer function (i.e., it does not produce energy) then $\hat{G}(s)$ is not necessarily passive. Mathematically the passivity condition is given by $G(i\omega) + G^*(i\omega) \geq 0$ or, for single-input single-output systems, the phase must be less than $\pm 90^\circ$). This technical point turns out to be very important since it is desired to fabricate equipment models using passive mechanical elements. The issue can be resolved with a method of passivity preserving balanced truncation described in [5]. The performance (both theoretically and in practice) is not compromised by the modified technique. The approach is illustrated below for two example systems.

EXAMPLE 1

A single-input single-output system was used to compare the performance obtained through balanced and modal truncation. The system is depicted in Fig. 1. An equipment model possessing 36 modes (72 states) is attached to a master structure with a fixed-base frequency normalized to unity. The equipment modes possess a uniform random frequency distribution in the interval [0.6,1.4]. Their magnitudes correspond to a uniform distribution in the interval [0,1]. This equipment model was selected to reflect the ambiguity encountered when attempting a modal reduction of equipment possessing moderate modal density.

Using the attachment point force and velocity as the input and output, respectively, the equipment model was reduced to four modes (8 states) using balanced and modal truncation. (The latter was carried out using $\left\| C_i B_i \right\| / \left| \zeta_i \omega_i \right|$ to rank the contribution of each mode to transfer function error.) With this choice of input and output, the transfer function corresponds to mechanical admittance. The admittance magnitude for the entire system (equipment and master structure) is plotted in Fig. 2. This quantity can be interpreted as the response of the master structure to a vertical disturbance force. From this plot, it can be seen that balanced truncation provides a better match to the full system in the neighborhood of the master structure's fixed-base frequency. While not depicted, this is also true of admittance phase angle. Furthermore, balanced truncation provides better low and high frequency amplitude matching.

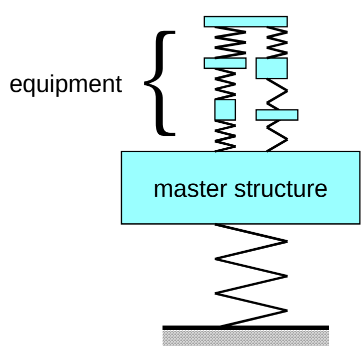

Fig. 1. One dimensional example system. The equipment model has 36 modes (72 states) uniformly distributed about the fixed-base frequency of the master structure.

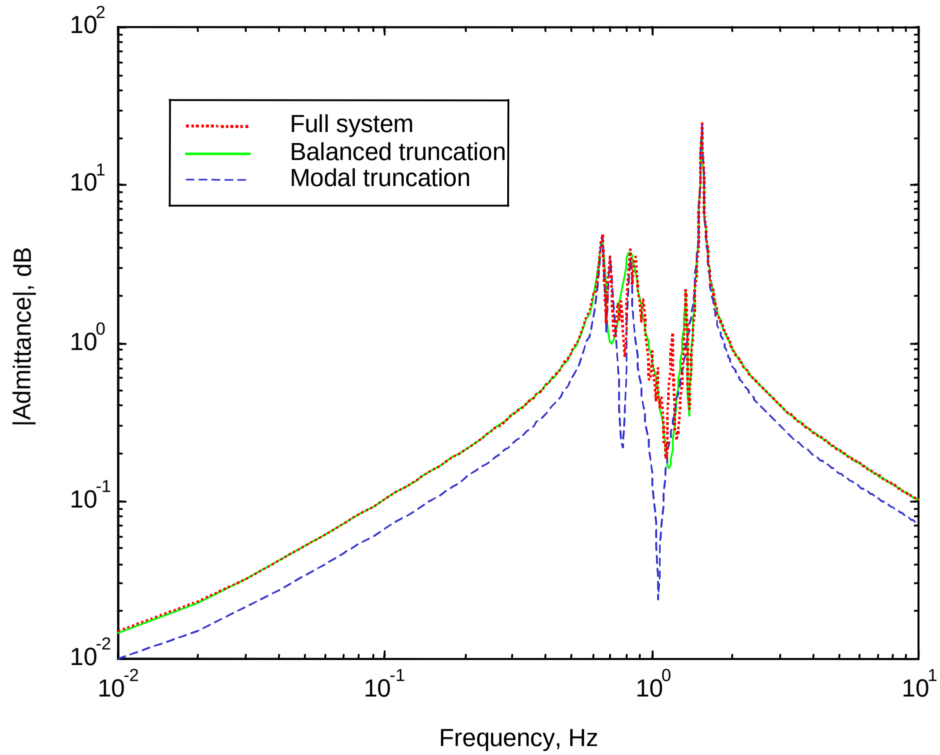

Fig. 2. Admittance (velocity/force) magnitude plots for systems consisting of the master structure and the full, balanced or modal equipment models.

To compare the truncation techniques in terms of shock severity, the shock spectrum of the master structure was computed in response to a half-sine force pulse of duration 0.5 seconds. The result is shown in Fig. 3. It can be seen that the modal truncation model causes the shock spectrum to be underestimated in the neighborhood of the master structure's fixed-base frequency by a factor of 0.52 to 0.67. This indicates that the modally truncated system absorbs significantly more energy in this frequency range than does the actual equipment. To a lesser degree, this model also underpredicts the shock spectrum at high frequencies. A shock trial conducted with such a model would erroneously suggest that the master structure provides a much safer shock environment than in actuality. In contrast, the equipment model obtained by balanced truncation provides good shock spectrum matching over the entire frequency range.

EXAMPLE 2

A two-dimensional finite element cabinet model was developed in order to test the passive, balanced realization technique for structures of more realistic complexity. The cabinet is constructed of beam elements and consists of three main compartments, with stringers and masses intended to simulate internally mounted components. The model appears in Fig. 4. There are two inputs and two outputs consisting of the vertical forces and velocities, respectively, measured at drive points 1 and 2. The stringer stiffnesses and masses were randomly chosen, so that a number of modes involving motion of the stringers and cabinet structure lie in the range of 15-40 Hz.

The full finite element model of the cabinet has 24 states, which through passive balanced truncation, is reduced to 4. The two drive point admittance magnitudes (diagonal elements of $G(s)$) are shown in Fig. 5 and Fig. 6. As was the case for the one-dimensional example, both low and high frequency asymptotic behavior is well captured by the truncation. In addition, the major dynamic effects are captured for all intermediate frequencies.

To confirm these observations, the cabinet model was mounted on a master structure consisting of a simply supported beam. The beam properties were chosen such that the modal frequencies and admittance amplitudes were comparable to those of the cabinet at the attachment points. The velocity response of the combined system to an impulsive force applied at attachment point 1 is shown in Fig. 7. The system incorporating the truncated equipment model accurately predicts both early and late time response. The shock spectrum at attachment point 1 of the combined system is plotted in Fig. 8 for a half-sine force pulse applied at this point. Excellent agreement between the full and reduced systems is observed.

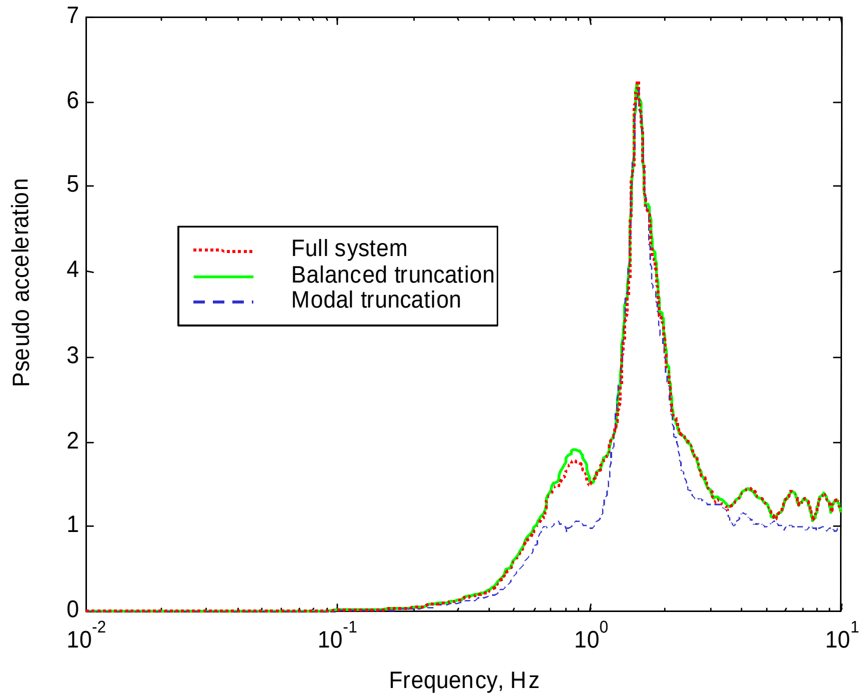

Fig. 3. Shock spectrum with 5% damping comparing response of the master structure to a half-sine force pulse for the full and truncated equipment models.

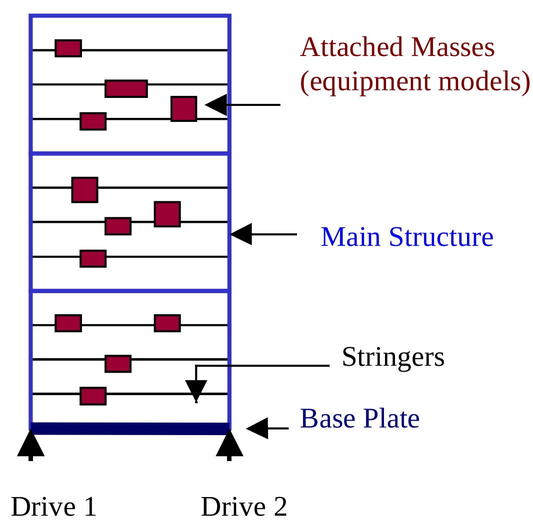

Fig. 4. Two-dimensional cabinet model. The two inputs and two outputs consist of the vertical drive point forces and velocities, respectively.

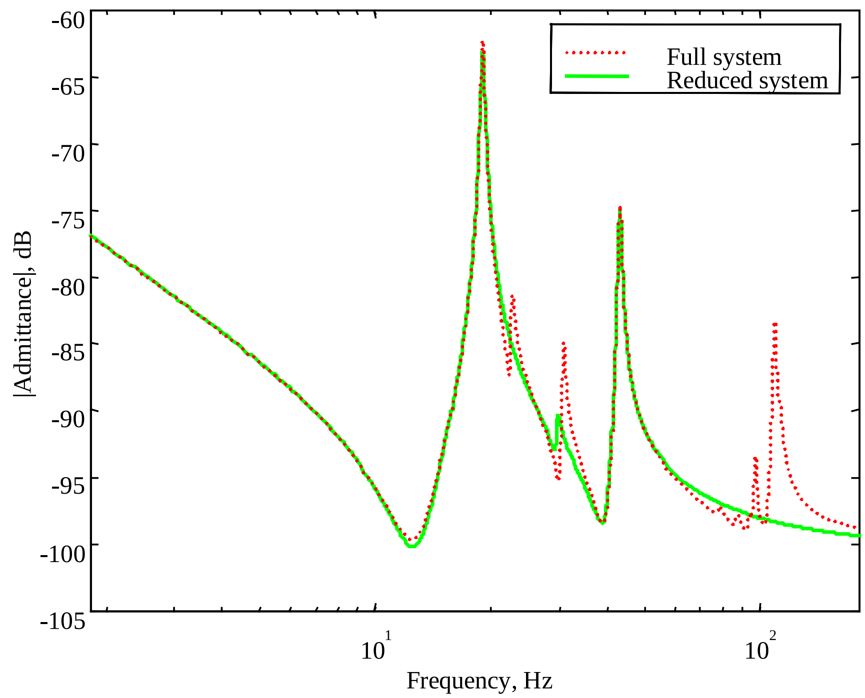

Fig. 5. Drive point admittance (velocity/force) at attachment point 1 for the full-order and reduced-order systems.

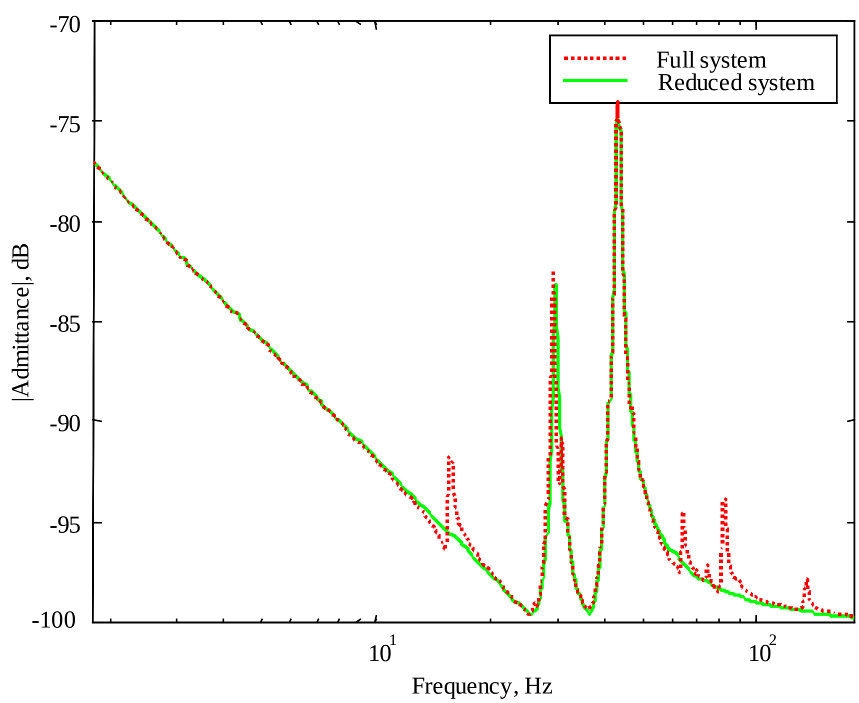

Fig. 6. Drive point admittance (velocity/force) at attachment point 2 for the full-order and reduced-order systems.

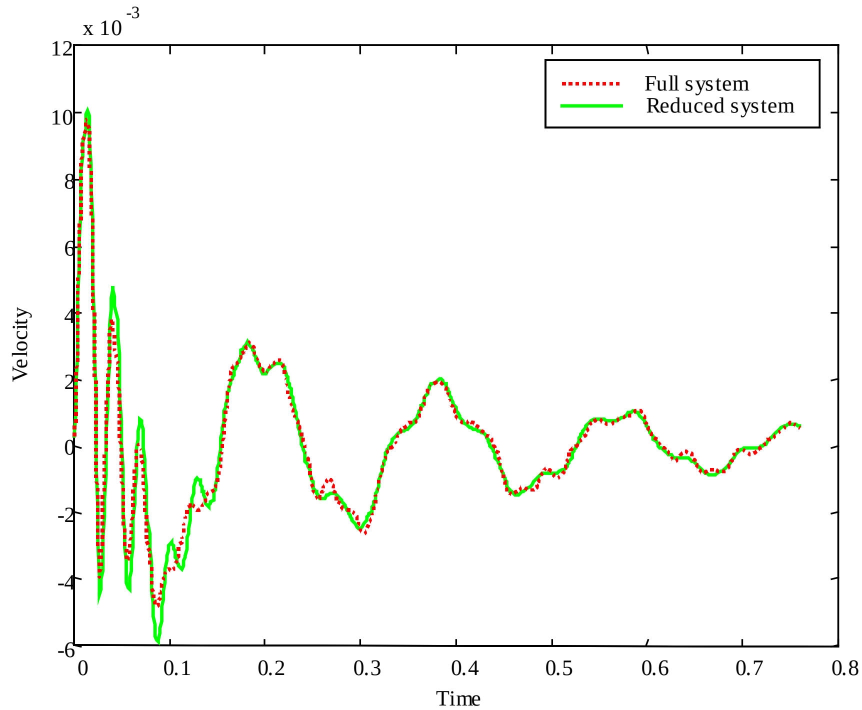

Fig. 7. Deck velocity at attachment point 1 in response to an impulsive force input.

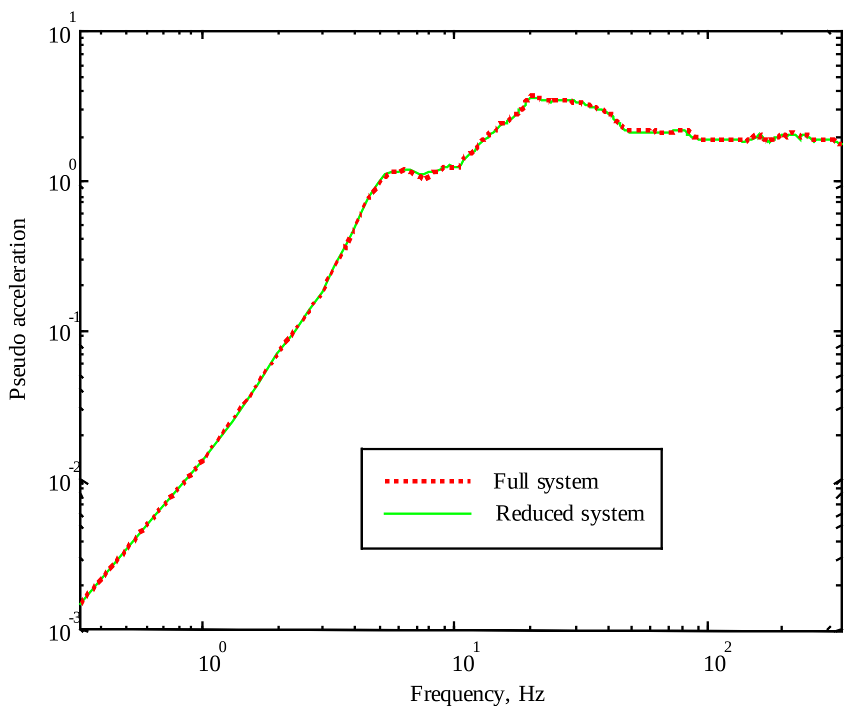

Fig. 8. Shock spectrum (5% damping) at attachment point 1 for the full-order and reduced-order systems.

CONCLUSIONS

Balanced truncation has been demonstrated as an effective alternative to modal truncation for the design of numerical or scaled mechanical equipment models. The method is inspired by the desire to minimize input-output error energy described in terms of 2-norms. The relationship between model complexity and error energy can be ascertained using known error bounds that depend on the truncated states. A simple modification makes it possible to ensure that the reduced system preserves the passivity of the full-order model.

The application of the approach to two example systems indicates that balanced truncation is effective from the viewpoints of the time, frequency and shock spectra domains. In addition, the first example established that balanced truncation may be superior to modal truncation in the context of shock loading. Additional work is needed to fully understand the analytical and practical implications for the emulation of actual equipment. Furthermore, for multi-input multi-output systems, no systematic technique exists for fabricating a reduced model obtained either through modal or balanced truncation. These issues are topics of current study.

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