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Stability of Rigid-body Dynamics with Sliding Frictional Contacts*

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Abstract

The use of rigid body models during frictional contact is often justified by proving the existence of a unique solution to the forward dynamic equations. The implicit assumption here is that the contact forces so obtained are stable. In this paper, the rigid-body assumption is relaxed and body-to-body contacts are modeled using springs and dampers. A singular perturbation analysis reveals additional necessary conditions to ensure contact force stability in the reduced rigid-body model. Furthermore, the analysis indicates that stability depends on a damping ratio associated with the rigid contacts.

1 Introduction

There are many robotic applications which necessitate the computation of the forward and inverse dynamics of constrained bodies. These include the automated planning of mechanical assembly tasks, dextrous manipulation, virtual reality systems, and in some cases, robotic grasping and parts fixturing as well. For the forward problem of simulation, the main goals are to compute system motion accurately and efficiently. In the case of inverse dynamics, a thorough understanding of the dynamic phenomena is needed in order to develop motion planning strategies.

Most often rigid body models are used to represent constrained systems. Of course, real bodies are compliant. Rigid bodies are reduced order models introduced for ease of computation. As such, their use must be justified. The singular perturbation method presented in this paper is a rigorous way to provide this justification (assuming, of course, that the complete order model is correct). This is in contrast to the common approach of justifying the rigid model by proving the existence and uniqueness of the rigid-body forward dynamics solution.

The analysis in this paper reveals that with sufficient normal-force-dependent friction (e.g., friction coefficients are sufficiently high), the normal contact forces become unstable and the rigid-body assumption is invalid. Only in certain cases is contact force instability manifested as existence and uniqueness problems of the rigid body forward dynamic equations. Other dynamic instabilities cannot be detected directly from the reduced-order rigid body models.

The paper is organized as follows. Related work is described in the next section. The following section describes the assumptions and formulates the dynamic equations. In section 4, the singular perturbation analysis is presented using lumped viscoelasticity at each friction contact. The validity of the rigid model is related to the stability of the boundary layer describing the contact forces. A criteria for deciding instability is given and is related to the existence and uniqueness result of the Linear Complementarity Problem (LCP) formulation. Section 5 summarizes the results.

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2 Literature Review

For frictionless systems, proofs of forward dynamics existence and uniqueness for rigid body systems can be found in dynamics texts. When small amounts of normal-force-dependent friction are included, experience has shown that the rigid-body equations remain well behaved. However, starting with Painlevé, examples of rigid-body systems with Coulomb friction were published which produced either no solution or multiple solutions to the forward dynamics problem [3,7,9,11,12].

The related literature addresses two issues. The first involves the detection or prediction of existence and uniqueness problems due to normal-force-dependent friction, e.g., [3,4,7, 9,11,12,15]. The second issue concerns modification of the system model in order to achieve an understanding and resolution of the forward dynamics existence and uniqueness problems.

The most general method for checking solution existence and uniqueness is that of casting the forward dynamics problem as an LCP in which unilateral contact constraints are described using complementarity constraints between the normal contact forces and the normal relative acceleration. This method was developed by Lötstedt for planar motion of systems with Coulomb friction at sliding unilateral contacts [7,8]. Generalizations to include rolling contacts and spatial motion have been proposed by Trinkle et al. [13]. For spatial motion, the complementarity formulation becomes nonlinear unless Coulomb's law is approximated.

Given applied forces or torques, one does not expect a mechanism to produce any of several possible motions or no motion at all. A failure to predict the actual motion is indicative of an inadequate model, an insufficient description of system state or both. The likely suspects are two: the rigid body assumption and the friction model.

While Coulomb's law is an extreme simplification of actual friction, any friction law depending on normal force could educe similar instabilities in rigid body mechanics. The authors are not aware of any attempts to resolve the existence and uniqueness problems solely through modification of the friction law.

In this paper, rigid bodies are considered as simplifications of bodies with compliant contacts. The normal force at each contact is modeled by a spring and damper in parallel. Consequently, the reaction forces, as functions of contact displacement and velocity, depend directly on system state.

We are not the first to consider models of this type [1,5,14,15]. Whitney used Hertzian contact models [15]. Wang et al. discretize the contact area into a collection of patches with lumped stiffness [14]. Howard et al. extend this work to enable transient analysis of contact forces and displacements given assumptions on the maximum friction forces [5]. Baraff used a damping/stiffness contact model to argue that the Principle of Constraints, which advocates the use of finite forces over impulses whenever possible, has no physical basis [1].

3 Constrained Rigid-body Dynamics

Consider an n degree of freedom system with generalized coordinates given by the vector q which is subject to m unilateral contact constraints. Let $\phi_i(q)$ be the minimum distance between the bodies comprising the ith contact. The unilateral contact constraint can be written as

$$\Phi_i(q) \geq 0 \tag{1}$$

with the equality holding only when the bodies are in contact.

The assumptions which are employed in the rest of the paper are as follows.

- Contact is defined by a finite number of point contacts.
- Contact normals are well defined and are linearly independent in the space of system generalized forces.
- Contact friction obeys Coulomb's law.
- The direction of the friction forces is known (always the case for planar motion).
- All contacts are sliding (not rolling).

Let the relative velocity in the tangential direction at contact i be $\Gamma_i(q)\dot{q}$ where $\Gamma_i^T \in \Re^n$.

The constraint distance functions can be written as a column vector, $\Phi(q) \in \Re^m$. Its Jacobian matrix, $\Phi_q = \partial\Phi/\partial q$, is used to project contact forces into the space of generalized forces and torques. In this way, the constrained dynamic equation can then be written as

$$M(q)\ddot{q} + h(q,\dot{q}) = u + \Phi_q^T(q,t)\lambda + f_f \tag{2}$$

where $M(q)$ is the inertia matrix and h consists of centrifugal, Coriolis and gravity terms. The generalized input forces and torques are given by u. λ is the vector of constraint force magnitudes in the normal direction and f_f is the sum of friction forces and torques expressed in terms of the generalized coordinates.

Following the Coulomb model, we assume friction depends linearly on the contact forces during sliding. Thus, the components of f_f can be written as

$$f_{fi} = \hat{\Phi}_{qi}^T(q,\dot{q},\mu_i)\,\lambda = -\sum_{i=1}^{m} \mu_i \,\mathrm{sgn}(\Gamma_i(q)\dot{q})\,\Gamma_i^T\,\lambda_i. \tag{3}$$

For convenience, we introduce the matrix $\hat{\Phi}_q \in \Re^{m\times n}$, whose rows are $\hat{\Phi}_{qi}$, $i = 1,\ldots,m$.

Collecting the coefficients of friction into the vector $\mu = [\mu_1,\mu_2,\ldots,\mu_m]^T$, the dynamic equation can be written as

$$M(q)\ddot{q} + h(q,\dot{q}) = u + \left[\Phi_q^T(q) + \hat{\Phi}_q^T(q,\dot{q},\mu)\right]\lambda \tag{4}$$

The vector of normal contact accelerations can be obtained by differentiating the vector Φ twice with respect to time.

$$\ddot{\Phi} = \Phi_q\ddot{q} + \dot{\Phi}_q\dot{q} \tag{5}$$

3.1 Formulation as a Linear Complementarity Problem

At a rigid, unilateral, sliding contact, the normal force, λ_i, and the normal acceleration, $\ddot{\Phi}_i$, form a complementary pair as they must satisfy

$$\lambda_i \geq 0, \quad \ddot{\Phi}_i \geq 0, \quad \lambda_i\ddot{\Phi}_i = 0 \tag{6}$$

Either the normal contact force is positive and the normal acceleration is zero or the normal acceleration is positive and the contact force is zero. Using (4) and (5), the vector of normal contact accelerations can be written in terms of the normal forces, λ, and the inputs, u.

$$\ddot{\Phi} = \Phi_q M^{-1}\left[\Phi_q^T + \hat{\Phi}_q^T\right]\lambda + \left(\Phi_q M^{-1}(u-h) + \dot{\Phi}_q\dot{q}\right) \tag{7}$$

In its general form, the linear complementarity problem is to solve for a vector $x \in \Re^m$ given the matrix $D \in \Re^{m\times m}$ and the vector $y \in \Re^m$ which satisfies

$$Dx + y \geq 0, \quad x \geq 0, \quad x_i(Dx+y)_i = 0 \tag{8}$$

The existence and uniqueness of x are described by the following definition and theorem from Cottle et al. [2].

Definition 3.1 *A matrix $D \in \Re^{m\times m}$ is a P-matrix if all of its principal minors are positive.*

Theorem 3.1 *A matrix $D \in \Re^{m\times m}$ is a P-matrix if and only if the LCP given by D and y has a unique solution x for all vectors $y \in \Re^m$.*

We can make the following identifications with the rigid body dynamics problem.

$$\begin{aligned} x &\approx \lambda \\ y &\approx \left(\Phi_q M^{-1}(u-h) + \dot{\Phi}_q \dot{q}\right) \\ D &\approx \Phi_q M^{-1}\left[\Phi_q^T + \hat{\Phi}_q^T\right] \end{aligned} \tag{9}$$

This result provides a necessary and sufficient condition on the matrix D for the existence and uniqueness of normal contact forces given arbitrary input forces and torques, u.

4 Singular Perturbation Analysis

While the LCP results discussed above provide conditions under which a solution to the rigid body forward dynamics problem with unilateral contacts exists and is unique, they do not directly address the stability of the rigid body solution. In this section, we assume a full-order model which includes contact compliance and damping and use a singular perturbation analysis to determine under what conditions the reduced-order rigid body model is stable and thus valid.

Of course, the correctness of the results depends upon the validity of the full-order model used in the analysis. For the purposes of this paper, we employ the simplest full-order model in which contact forces are a function of system state. In particular, we relax the rigid-body assumption by modeling the contact force using lumped stiffness and damping. The constraint equations, $\Phi = 0$, are satisfied exactly only when there is contact with zero normal force. Therefore, we can write λ as

$$\lambda_i = \begin{cases} -\left[C\dot{\Phi} + K\Phi\right]_i, & \Phi_i < 0 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

where $C, K \in \Re^{m\times m}$ are diagonal damping and stiffness matrices, respectively.

Apparent interpenetration corresponding to $\Phi < 0$ is justified physically as normal deformation of the contact asperities.

The damping and stiffness matrices are expressed in terms of a small parameter ϵ such that the rigid model is recovered in the limit $\epsilon \rightarrow 0$. For simplicity, stiffness and damping as well as ϵ are assumed the same at each contact with $c > 0$, $k > 0$ and $I \in \Re^{m\times m}$.

$$C = \frac{cI}{\sqrt{\epsilon}}, \quad K = \frac{kI}{\epsilon} \tag{11}$$

The contact forces are given by

$$\lambda_i = -\left[\frac{c}{\sqrt{\epsilon}}\dot{\Phi} + \frac{k}{\epsilon}\Phi\right]_i \geq 0 \tag{12}$$

This contact model was used by McClamroch to represent a manipulator constrained by a frictionless environment [10]. The relative scaling of C and K is chosen so as to preserve the damping ratio of the contact, $\zeta = \sqrt{c^2/k}/2$, in the limit. In an impact between two bodies with this contact model, a coefficient of restitution, $e = 1$, for an elastic collision corresponds to $c = \zeta = 0$. For an inelastic collision, $e = 0$ corresponds to $k < \infty$ and $\zeta = \infty$. To ensure separation of time scales, however, the stiffness term must dominate damping in the limit. Specifically, k and c must satisfy $k/c \gg \sqrt{\epsilon}$.

The resulting dynamic equation is given by

$$M(q)\ddot{q} + h(q,\dot{q}) = u - \left[\Phi_q^T + \hat{\Phi}_q^T\right]\left[\frac{c}{\sqrt{\epsilon}}\dot{\Phi} + \frac{k}{\epsilon}\Phi\right] \tag{13}$$

To reveal the two time scales of (13), it must be written in standard form which separates the fast and slow variables.

Following [10], the contact interpenetration distances are used as the fast variables, z, and the constrained rigid body modes are taken to be the slow variables, x. They can be written in terms of q as

$$\begin{bmatrix} \epsilon z \\ x \end{bmatrix} = \begin{bmatrix} \Phi(q_1, q_2) \\ q_2 \end{bmatrix} \in \Re^n \tag{14}$$

where q has been partitioned such that $q_1 \in \Re^m$ and $q_2 \in \Re^{(n-m)}$. Given our assumption on the linear independence of contact normal vectors, we know $\text{rank}(\Phi_q) = m$ and the inverse function theorem allows us to write

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \Omega(\epsilon z, x) \\ x \end{bmatrix} \tag{15}$$

To express (13) in the new coordinates, we write the invertible Jacobian of (15)

$$J(\epsilon z, x) = \begin{bmatrix} \frac{\partial\Omega}{\partial\epsilon z} & \frac{\partial\Omega}{\partial x} \\ 0 & I \end{bmatrix} \tag{16}$$

and obtain, with all quantities expressed in terms of $(\epsilon z, x)$ and their derivatives,

$$\begin{aligned} &J^T M J \begin{bmatrix} \epsilon\ddot{z} \\ \ddot{x} \end{bmatrix} + J^T M \dot{J} \begin{bmatrix} \epsilon\dot{z} \\ \dot{x} \end{bmatrix} + J^T h \\ &= -J^T\left(\Phi_q^T + \hat{\Phi}_q^T\right)\left(c\sqrt{\epsilon}\dot{z} + kz\right) + J^T u \end{aligned} \tag{17}$$

Since the Jacobian of

$$\Phi(q_1, q_2) = \Phi(\Omega(\epsilon z, x); x) = \epsilon z \tag{18}$$

is given by

$$\frac{\partial\Phi}{\partial(\epsilon z, x)} = \Phi_q J = [I \;\; 0] \tag{19}$$

its transpose, appearing in (17), is

$$J^T \Phi_q^T = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{20}$$

and rearranging we have

$$\begin{aligned} \begin{bmatrix} \epsilon\ddot{z} \\ \ddot{x} \end{bmatrix} &= -J^{-1}\dot{J}\begin{bmatrix} \epsilon\dot{z} \\ \dot{x} \end{bmatrix} - J^{-1}M^{-1}h \\ &\quad - J^{-1}M^{-1}J^{-T}\begin{bmatrix} c\sqrt{\epsilon}\dot{z} + kz \\ 0 \end{bmatrix} \\ &\quad - J^{-1}M^{-1}\hat{\Phi}_q^T\left(c\sqrt{\epsilon}\dot{z} + kz\right) + J^{-1}M^{-1}u \end{aligned} \tag{21}$$

Defining the notation

$$(A)_{uv} = [a_{ij}],\ i = 1, \ldots, u;\ j = 1, \ldots, v \tag{22}$$

the fast dynamics can be written as

$$\begin{aligned} \epsilon\ddot{z} \quad = \quad & -\left(J^{-1}\dot{J}\right)_{mn} \begin{bmatrix} \epsilon\dot{z} \\ \dot{x} \end{bmatrix} - \left(J^{-1}M^{-1}\right)_{mn} h \\ & - \left(J^{-1}M^{-1}J^{-T}\right)_{mm} \left(c\sqrt{\epsilon}\dot{z} + kz\right) \\ & - \left(J^{-1}M^{-1}\hat{\Phi}_q^T\right)_{mm} \left(c\sqrt{\epsilon}\dot{z} + kz\right) \\ & + \left(J^{-1}M^{-1}\right)_{mn} u \end{aligned} \tag{23}$$

(21) is in standard form if (23) has one or more isolated real roots, $\bar{z}(x, \dot{x})$, when evaluated at $\epsilon = 0$ [6]. The reduced, rigid-body model $(\bar{x}, \dot{\bar{x}})$ is obtained from the last $n - m$ rows of (21) through substitution of $\bar{z}$ and $\epsilon = 0$. (23) can be solved for $\bar{z}$ as long as the matrix sum $\left(J^{-1}M^{-1}J^{-T}\right)_{mm} + \left(J^{-1}M^{-1}\hat{\Phi}_q^T\right)_{mm}$ is invertible. Furthermore, it must be the case that $\bar{z}_i \leq 0$ to satisfy (12) and (14) for unilateral constraints. The ability to solve for $\bar{z}$, however, tells us nothing of the stability of this point and consequently nothing of the stability of the rigid model.

To study stability of the boundary layer, the change of coordinates $y = z - \bar{z}$ is introduced along with a stretched time scale, $\tau = t/\sqrt{\epsilon}$. Since in the time scale τ, the quantities t and x are slowly varying, (23) can be evaluated at $\epsilon = 0$. With the elimination of terms associated with the steady solution, $\bar{z}$, the boundary layer dynamics, with equilibrium at $y = 0$, are described by

$$y'' + \left[\left(J^{-1}M^{-1}J^{-T}\right)_{mm} \left(J^{-1}M^{-1}\hat{\Phi}_q^T\right)_{mm}\right](cy' + ky) = 0 \tag{24}$$

where $'$ indicates differentiation with respect to τ. Recalling the form of J from (16), its inverse is

$$J^{-1}(\epsilon z, x) = \begin{bmatrix} \left[\frac{\partial\Omega}{\partial\epsilon z}\right]^{-1} & -\left[\frac{\partial\Omega}{\partial\epsilon z}\right]^{-1}\left[\frac{\partial\Omega}{\partial x}\right] \\ 0 & I \end{bmatrix} \tag{25}$$

and the matrix products in (24) involve only the first m rows of this matrix. Note that the Jacobian of (14) is the inverse of $J(\epsilon z, x)$.

$$\frac{\partial(\epsilon z, x)}{\partial(q_1, q_2)} = \begin{bmatrix} \frac{\partial\Phi}{\partial q_1} & \frac{\partial\Phi}{\partial q_2} \\ 0 & I \end{bmatrix} = J^{-1}(\epsilon z, x) \tag{26}$$

Equating (25) and (26), we have

$$\begin{aligned} \left(J^{-1}\right)_{mn} &= \begin{bmatrix} \left[\frac{\partial\Omega}{\partial\epsilon z}\right]^{-1} & -\left[\frac{\partial\Omega}{\partial\epsilon z}\right]^{-1}\left[\frac{\partial\Omega}{\partial x}\right] \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial\Phi}{\partial q_1} & \frac{\partial\Phi}{\partial q_2} \end{bmatrix} = \Phi_q \end{aligned} \tag{27}$$

Thus, we can express the boundary layer system in the original coordinates.

$$y'' + \left[\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)\right](cy' + ky) = 0 \tag{28}$$

or

$$\begin{aligned} \begin{bmatrix} y' \\ y'' \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -kD & -cD \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \\ D &= \left[\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)\right] \end{aligned} \tag{29}$$

Our result can be stated as the following theorem.

Theorem 4.1 (Validity of Rigid Body Model) *Consider the system described by (13), which can also be written as (21), with initial conditions that are smooth functions of ϵ. If the following conditions are satisfied in a neighborhood of $(x, \dot{x}, z, \dot{z})$ for all time $t \in [0, t_1]$*

1. *the terms on the right hand side of (21) and their first partial derivatives are bounded and continuous,*
2. *the origin of the boundary layer system is exponentially stable, and*
3. *$\bar{z}(x, \dot{x})$ has continuous first partial derivatives with respect to its arguments*

then the following are true

- *the reduced rigid-body model, $(\bar{x}, \dot{\bar{x}})$, obtained from the last $n - m$ rows of (21) by substitution of $\epsilon = 0$ and $z = \bar{z}$, has a unique bounded solution for all $t \in [t_0, t_1]$ where $t_0 \in [0, t_1)$, and*
- *there exist positive constants δ and ϵ^* such that for all initial conditions $z(t_0, \epsilon)$, $\dot{z}(t_0, \epsilon)$, $x(t_0, \epsilon)$ and $\dot{x}(t_0, \epsilon)$ satisfying*

$$\left\|\begin{bmatrix} z(t_0, 0) \\ \dot{z}(t_0, 0) \end{bmatrix} - \begin{bmatrix} \bar{z}(t_0, x(t_0, 0), \dot{x}(t_0, 0)) \\ 0 \end{bmatrix}\right\| < \delta \tag{30}$$

and $0 < \epsilon < \epsilon^*$, on the interval $[t_0, t_1]$, the singular perturbation problem has a unique solution $x(t, \epsilon)$, $\dot{x}(t, \epsilon)$, $z(t, \epsilon)$, $\dot{z}(t, \epsilon)$ and

$$\begin{bmatrix} x(t, \epsilon) \\ \dot{x}(t, \epsilon) \end{bmatrix} - \begin{bmatrix} \bar{x}(t) \\ \dot{\bar{x}}(t) \end{bmatrix} = O(\sqrt{\epsilon}) \tag{31}$$

$$z(t, \epsilon) - \bar{z}(t) = O(\sqrt{\epsilon}) \tag{32}$$

Proof: The first result follows from conditions *1* and *2*. A necessary condition for the boundary layer to be stable is that D be invertible. Thus, $\bar{z}$ and the corresponding λ exist, and are unique for all bounded inputs. Together with *1*, standard existence and uniqueness results for differential equations yield the first result for $\bar{x}$. The second result follows directly from Tikhonov's theorem. See, e.g., Theorem 8.1 in [6]. □

In the formulation of this result, the normal acceleration vector, $\ddot{\phi}$, was set to zero. Consequently, theorem 4.1 applies directly to bilateral constraints in which the reaction forces can be of either sign.

However, the theorem also applies to all active unilateral constraints. For a given input vector, the problem of solving for the active constraints such that (6) is satisfied is nothing more than solving the LCP. Therefore, in solving the forward dynamics problem, we must first solve the LCP to determine the active constraints and then use theorem 4.1 to check if they are stable.

Since we expect that, during many motions, a set of constraints will remain active for some time interval, we would like to know whether or not the LCP existence and uniqueness result implies the result of the preceding theorem or vice versa. The main constraint of the theorem is the requirement for stability of the boundary layer. In the following section, the conditions for boundary layer stability are developed and compared with the LCP result of theorem 3.1.

4.1 Stability of Boundary Layer

By linearization at the origin, stability of the boundary layer depends on the eigenvalues of $D = \Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$ evaluated at $\bar{z}(x, \dot{x})$ as well as the values of $c > 0$ and $k > 0$.

Remark 4.1 *The matrix D which, along with c and k, determines stability of the boundary layer is the same matrix used to determine solution existence and uniqueness in the rigid-body LCP formulation.*

In what follows, we show that while the LCP formulation requires only that D be a P-matrix to ensure solution existence and uniqueness, the singular perturbation analysis imposes additional constraints on the D associated with active constraints.

Given that the columns of $\hat{\Phi}_q^T$ depend linearly on the friction coefficient of the associated contact, the frictionless boundary layer depends on the eigenvalues of $\Phi_q M^{-1}\Phi_q^T$. Since M is a positive definite, symmetric matrix, $\Phi_q M^{-1}\Phi_q^T$ is as well and the frictionless boundary layer is always stable. As the friction coefficients are increased from zero, the eigenvalues of D can migrate from their starting values on the positive real axis.

In the most general case, the matrix may be defective and the most a coordinate transformation of (28) will achieve is the Jordan form, J, of $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$. This transformation leads to the characteristic equation

$$\det(Is^2 + cJs + kJ) = 0 \tag{33}$$

where $I \in \Re^{m\times m}$ is the identity matrix. The determinant of a triangular matrix is given by the product of the diagonal elements which are of the form

$$s^2 + c(a + ib)s + k(a + ib) = 0 \tag{34}$$

where $a + ib$ is a complex eigenvalue of $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$. The boundary layer is stable if (34) has stable roots for all eigenvalues of $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$.

Examining (34), it is easy to show that for exponential stability

$$\begin{gathered} a > 0 \\ ac^2(a^2 + b^2) - b^2 k > 0 \end{gathered} \tag{35}$$

This leads to the following theorem.

Theorem 4.2 *Only in the case of a single sliding contact can we say that the LCP solution exists and is unique if and only if the boundary layer is exponentially stable for any $\zeta > 0$.*

Proof: For a single contact, $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$ is a scalar and the result follows from (35) and from the fact that the real eigenvalues of P-matrices must be positive [2]. For multiple contacts, if $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$ is a P-matrix, it can possess complex eigenvalues with positive or negative real parts. (Complex eigenvalues with negative real parts can occur for P-matrices in $\Re^{3\times 3}$ or higher. See Cottle et al. for an example [2].) If the real part of an eigenvalue is negative, the boundary layer is unstable. If the real part of an eigenvalue is positive, boundary layer stability depends on the particular value of $\zeta = \sqrt{c^2/k}/2$. In either case, boundary layer instability corresponds to contact forces which are exponentially growing sinusoids. Finally, we demonstrate that boundary layer stability does not imply LCP existence and uniqueness. The following matrix has eigenvalues 1 and 3 which would produce a stable boundary layer for any $\zeta > 0$, however, it is clearly not a P-matrix. □

$$\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix} \tag{36}$$

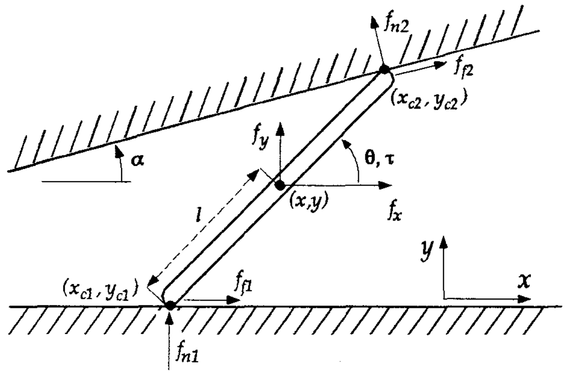

Figure 1: Planar rod of mass m and moment of inertia J in contact with two immobile walls.

4.2 Example

The results of theorems 4.1 and 4.2 are illustrated by the planar motion of a rod in contact with two walls depicted in Figure 1. The rigid rod possesses three degrees of freedom which can be described by the x and y coordinates of its center of mass and by its angular orientation, θ. Thus, we take $q = [x, y, \theta]^T$. The rod has a mass, m, and a moment of inertia, J, about its mass center. This is analogous to geometries studied in [12,15].

There are two constraints which are given by

$$\Phi = \begin{bmatrix} y - l\sin\theta \\ (x + l\cos\theta)\sin\alpha - (y + l\sin\theta)\cos\alpha \end{bmatrix} = 0 \tag{37}$$

This can be differentiated to obtain

$$\Phi_q = \begin{bmatrix} 0 & 1 & -l\cos\theta \\ \sin\alpha & -\cos\alpha & -l\cos(\theta - \alpha) \end{bmatrix} \tag{38}$$

The Coulomb friction forces at the contacts are given by

$$f_f = \hat{\Phi}_q^T \lambda \tag{39}$$

where the signs of λ's components to maintain contact are taken to be positive and

$$\hat{\Phi}_q^T = \begin{bmatrix} -\mu_1\ \mathrm{sgn}(\dot{x}_{c1}) & -\mu_2\ \mathrm{sgn}(\dot{x}_{c2})\cos\alpha \\ 0 & -\mu_2\ \mathrm{sgn}(\dot{x}_{c2})\sin\alpha \\ -\mu_1\ \mathrm{sgn}(\dot{x}_{c1}) l \sin\theta & \mu_2\ \mathrm{sgn}(\dot{x}_{c2}) l \sin(\theta-\alpha) \end{bmatrix} \tag{40}$$

where $\dot{x}_{ci}$ denotes the x component of rod velocity at contact i.

In this case, $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right) = [a_{ij}]$ where

$$\begin{aligned} a_{11} &= m^{-1} + (l^2/J)(\cos^2\theta + \mu_1\ \mathrm{sgn}(\dot{x}_{c1})\sin\theta\cos\theta) \\ a_{12} &= -(\cos\alpha + \mu_2\ \mathrm{sgn}(\dot{x}_{c2})\sin\alpha)/m \\ &\quad +(l^2/J)\cos\theta(\cos(\theta-\alpha) \\ &\quad -\mu_2\ \mathrm{sgn}(\dot{x}_{c2})\sin(\theta-\alpha)) \\ a_{21} &= -(\cos\alpha + \mu_1\ \mathrm{sgn}(\dot{x}_{c1})\sin\alpha)/m \\ &\quad +(l^2/J)\cos(\theta-\alpha)(\cos\theta + \mu_1\ \mathrm{sgn}(\dot{x}_{c1})\sin\theta) \\ a_{22} &= m^{-1} + (l^2/J)\cos(\theta-\alpha)(\cos(\theta-\alpha) \\ &\quad -\mu_2\ \mathrm{sgn}(\dot{x}_{c2})\sin(\theta-\alpha)) \end{aligned} \tag{41}$$

For a given rod and wall configuration described by (θ, α), the eigenvalues of $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$ are determined by the dimensionless group $1 \le ml^2/J < \infty$.

Consider the case when $\alpha = \pi/4$, $\theta = \pi/2.3$ (78°) and $ml^2/J = 3$ (uniform distribution of mass). The stability map for this case appears in Figure 2. The two solid curves labeled $c^2/k = 0$ mark the boundaries of unstable regions containing complex eigenvalues with positive real parts. These eigenvalues produce contact forces which are exponentially growing sinusoids. As contact damping, described by $4\zeta^2 = c^2/k$, is increased, the stable region expands to include portions of these regions as shown by dashed lines.

There is a small unstable region labeled at the top of the graph which contains complex eigenvalues with negative real parts. For large values of c^2/k, the region of instability is reduced to the union of the two regions with $a < 0$. As can be seen from the figure, boundary layer instabilities can occur for friction coefficients typical of unlubricated contact.

Figure 3 shows the region in which $\Phi_q M^{-1}\left(\Phi_q^T + \hat{\Phi}_q^T\right)$ is a P-matrix. In this region, zero, one or both contacts may be maintained depending on the value of the input vector. In all cases, a unique solution exists. The region in which the P-matrix conditions are not satisfied is composed of three subregions which each correspond to at least one of the three principal minors being negative. In these regions, certain input vectors may produce multiple solutions to the LCP while for others, no solutions exists.

This example clearly demonstrates the result of Theorem 4.2. Boundary layer stability does not imply that the P-matrix condition is met nor does the latter imply the former.

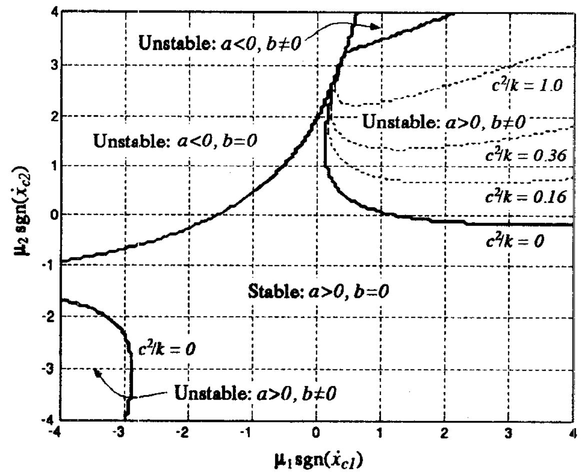

Figure 2: Stability map for rod in contact with two walls. $\theta = \pi/2.3$ (78°), $\alpha = \pi/4$, $ml^2/J = 3$ (uniform distribution of mass). Stability of the boundary layer is indicated in terms of eigenvalues $a + ib$ of matrix D. Region marked stable is marginally stable for $c^2/k = 0$ and exponentially stable for $c^2/k > 0$. Boundary of stable region is shown for several values of c^2/k.

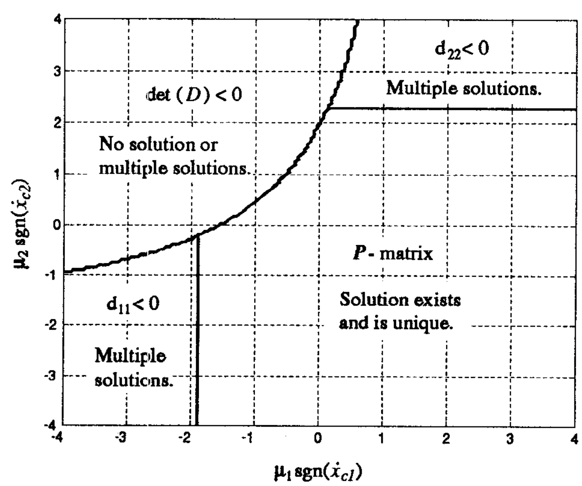

Figure 3: P-matrix map for rod in contact with two walls. $\theta = \pi/2.3$ (78°), $\alpha = \pi/4$, $ml^2/J = 3$ (uniform distribution of mass). In the regions where one or more principal minors are negative, some input vectors do produce unique solutions.

5 Conclusion

Using singular perturbation theory, justification of the reduced (rigid body) model requires both the existence and the stability of the solution. LCP theory provides a means to determine whether or not a unique solution to the unilateral contact problem exists for all values of input. It does not, however, reveal the stability of the solution. The contribution of this paper is to elucidate the conditions under which the contact forces at the active constraints are stable. Surprisingly, even as the rigid limit is approached, contact force stability depends on the damping ratio of the contacts – a quantity usually only indirectly associated with rigid bodies through the coefficient of restitution.

Testing the validity of the rigid body problem during simulation involves three steps. First, LCP results can be used to confirm the existence and uniqueness of the contact forces. Next, given the input vector, the LCP can be solved to determine the active constraints. Finally, theorem 4.1 can be employed to verify the stability of the active constraint forces. During those trajectory segments in which the set of active constraints is constant, it would be convenient to apply a single test to decide both (i) the existence and uniqueness of the solution as well as (ii) its stability. Only for a single contact is such a test available since, in this case, the sets of configurations satisfying (i) and (ii) are identical. For multiple contacts, these sets at most intersect and the existence of a simple test to decide membership in both is an open question.

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