Avoiding Stick-Slip Through PD Control

Pierre E. Dupont

Abstract—**This note addresses the question of how to achieve steady motion at very low velocities using proportional-derivative (PD) control. Most prior work in control has used friction models which depend only on the current value of velocity. This type of analysis indicates that stick-slip can be avoided only through velocity feedback. The tribology literature, however, indicates that friction also depends on the history of motion. By including this dependence, a second regime of stable motion is revealed which is associated with position feedback gains above a critical value. Two experimentally-based dynamic friction models are compared using a linearized stability analysis. In accord with experiment, as state variable friction model exhibits asymptotically stable motion for any system stiffness (position feedback gain) exceeding a critical value. This property is not exhibited by a time-lag friction model.**

I. Introduction

The effect of friction on machine performance has been demonstrated by a number of researchers [1], [3], [7], [15]. Its effect is most noticeable at very low velocities. At these speeds, motion tends to be intermittent. Commonly referred to as stick-slip, intermittent motion can lead to overshoot and large-amplitude position or force limit cycling.

There are several control approaches which have proven effective in ameliorating the effects of friction at low velocities. These are high-gain proportional-derivative (PD) control [1], model-based feedback or feedforward compensation [1], [3], [4], and impulsive control [16]. High-gain PD control is among the oldest techniques. It has been experimentally observed for many years that sufficiently stiff and/or damped systems do not exhibit stick-slip [10]. A complete analytical explanation of these observations has been lacking in the controls community, however, due to the tendency to employ friction models which depend only on the current value of velocity.

The existence of history-dependent friction behavior has been known to the tribology community for 50 years [13], and the missing analytical explanation has been known to the rock mechanics community for a decade [11]. Ongoing experimental work by the author [8], as well as related work within the tribology [9] and geophysics [6] communities, has shown that the friction transient effects which underpin the analytical result for rocks are also present in lubricated engineering materials. The purpose of this paper is to cast this stability result in the framework of machine control and to compare two experimentally-based dynamic friction models.

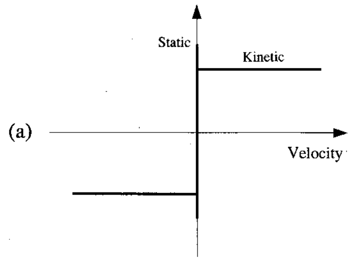
A. Problem Definition: Steady Sliding

Typically, machines which exhibit stick-slip possess a minimum stable velocity below which the stick-slip occurs. To design a PD controller for low-velocity motion, there are two possible approaches. In the first approach, one can ask what gains make the stick-slip limit cycle unstable so that the system will converge asymptotically to steady sliding. In the second approach, one can look for gains which provide a reasonably-sized domain of attraction so that the equilibrium point of steady sliding remains attractive and asymptotically stable under expected perturbations. The answers to these questions depend critically on the friction model.

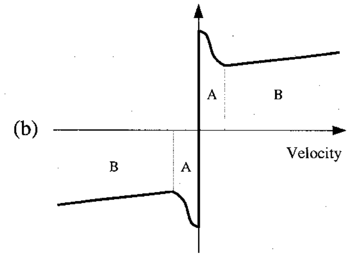
Manuscript received July 3, 1992; revised March 10, 1993. This work was supported in part by IBM Corp. through a postdoctoral fellowship and by the U.S. Army Research Office under Grant DAAL03-89-K-0112.

The author is with the Department of Aerospace and Mechanical Engineering, Boston University, Boston, MA 02215 USA.

IEEE Log Number 9400355.



(a)



(b)

Fig. 1. Friction force versus velocity: (a) kinetic/static model; (b) typical curve for hard materials separated by liquid lubricants.

The first approach is more difficult because it requires knowledge of friction behavior during sticking. The second approach is the one taken in this paper. It is a useful case to study because, through a linearized analysis, it can provide practical lower bounds on the PD gains necessary for stability. Only constant-velocity trajectories will be considered explicitly here. This analysis can be extended to more general nonsticking trajectories by selecting set points with intersecting domains of attraction.

In the next section, friction modeling is discussed. The state variable and time delay friction models, which include history dependence, are introduced. In Section III, a simple system model is presented along with a stability analysis for each friction model. Their stability criteria are compared with those obtained using a friction model which depends only on current velocity. Section IV discusses the implications of this analysis for the control of steady, low-velocity motion.

II. Friction Modeling

The characteristic friction-velocity curve for hard materials separated by liquid lubricants is contrasted with the static-kinetic model in Fig. 1. For liquid lubricants, this curve, with a rescaling of the x-axis, is usually referred to as the Stribeck curve. The accepted reasoning for its shape is as follows. At very low velocities, the liquid is squeezed from between the bodies and the shearing of solid-to-solid contacts plays a dominant role. At slightly higher velocities, pockets of lubricant under high pressure bear part of the load. At sufficiently high speeds, a fluid film separates the solids and bears the entire load. The friction curve is positively sloped in this regime. It is the negative slope of region A in Fig. 1(b) which promotes instability.

Mathematical models of the friction-velocity curve of Fig. 1(b) often represent the transition from static to kinetic fraction by a term exponential in velocity [1], [4]. A general model of this form including Coulomb and viscous friction is

$$f(V) = [c_0 + c_1|V| + c_2 e^{-(|V|/c_3)^{c_4}}]\,\mathrm{sgn}\,(V) \tag{1}$$

with constants $c_0, \cdots, c_4 > 0$.

It is important to note that friction-velocity curves represent the behavior of steady sliding. Sampson *et al.* where among the first to note the multivalued behavior of friction [13]. Friction is not determined by current velocity alone; it also depends on the history of motion. This functional relationship for the friction, f, can be expressed as

$$f(t) = \mathcal{F}[V(\tau), \sigma_n(\tau)], \quad -\infty < \tau < t \tag{2}$$

in which V denotes velocity and σ_n denotes normal stress.

In a control context, Dahl [5] and Walrath [15] developed history-dependent models for the friction and stiffness in ball bearings oscillating about zero velocity. Their steady-state friction-velocity curves, however, were flat. They observed a constant Coulomb friction level dependent only on the sign of velocity. Their models' transient response resulted only from velocity reversals. This behavior is typical of gimbals used in pointed and tracking applications. In most machines, bearing friction is negligible and other sources, such as transmission elements, dominate. In these cases, it has been demonstrated that steady-state friction is similar to Fig. 1(b) [1], [4]. Thus, the Dahl model does not appear to be appropriate for general applications.

A. State Variable Model

Independent of the tribology community, researchers in earthquake prediction suggested that earthquakes are stick-slip events due to the relative motion of the earth's crustal plates. As a result, Ruina and others have worked on the experimental and theoretical development of constitutive friction relations [11], [12]. These relations are referred to as state variable friction models. Recently, the experimental behavior upon which these models are based has been observed for engineering materials such as teflon on steel [6] and by the author for lubricated steel on steel [8].

A simple model of this type for velocity, $V > 0$, and constant normal stress is [12]

$$f(t) = f_0 + A\ln(V(t)/V_0) + \theta(t) \tag{3}$$

$$\dot{\theta}(t) = -\frac{V(t)}{L}[\theta(t) + B\ln(V(t)/V_0)] \tag{4}$$

in which θ is the scalar state variable and L is the characteristic length controlling the evolution of θ. The pair (V_0, f_0) corresponds to any convenient point on the steady-state friction-velocity curve. In this case, the steady-state curve is given by

$$f_{ss}(V) = f_0 + (A - B)\ln(V/V_0). \tag{5}$$

The functional form of this and other state variable models was deduced experimentally. In these experiments, the response of the friction force to small changes in steady velocity was measured. Velocity was precisely controlled by a closed-loop system having a very fast dynamic response, faster than that of the friction transient response [6], [8], [12]. Ruina's data, spanning several orders of magnitude of velocity, were more accurately represented by adding a second state variable to the model above [12].

B. Time-Delay Model

Additional evidence of the history-dependence of friction is provided by the experiments of Hess and Soom [9]. They measured friction-force response to an oscillating velocity of constant sign in a lubricated line contact under closed-loop control. The amplitude of the velocity oscillation was large enough to overlap most of region A and part of B in Fig. 1(b). They found that as the frequency of oscillation increases, the friction-velocity curve became a closed loop centered about the steady-state friction curve.

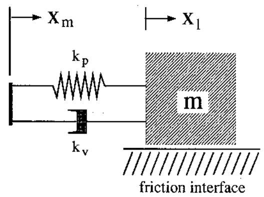


Fig. 2. Sliding system model. The input is motor velocity, $\dot{x}_m = V_0$, and the output is block velocity, $\dot{x}_l$.

Hess and Soom modeled this behavior by introducing a simple time delay in their steady-state equation

$$f(t) = c_0V(t-\alpha) + \frac{c_1}{1 + c_2V^2(t-\alpha)}, \quad V > 0 \tag{6}$$

where V is velocity, α is the time delay and c_0, c_1, c_2 are positive constants. Recently, a time-delay model has been used to study the stability of stick-slip limit cycles [2]. Note that the state variable models were specifically developed to model low velocity effects while steady-state models are formulated to fit the entire velocity operating range.

III. Linearized Friction Analysis

Consider the class of friction models which depend on the velocity history, assuming constant normal stress

$$f(t) = \mathcal{F}[V(\tau)], \quad -\infty < \tau \leq t. \tag{7}$$

Assume that f can be separated into an instantaneous rate-dependent component and an evolutionary component. The latter tends toward a steady-state value for sufficient displacement at a particular velocity. A friction law of this type, linearized about velocity V_0, can be written

$$f(t) = f_{ss}[V_0] + f_vV(t) - \int_0^t g(t-\tau)V(\tau)\,d\tau \tag{8}$$

in which f_v and $g(t)$ are dependent on V_0. From this equation, the instantaneous rate of change of friction with velocity is given by

$$\frac{\partial f(V)}{\partial V} = f_v \geq 0 \tag{9}$$

and the steady-state change of friction with velocity is

$$\frac{df_{ss}(V)}{dV} = f_v - \int_0^\infty g(t)\,dt < 0 \tag{10}$$

where the signs of these terms correspond to those observed experimentally in the unstable low-velocity regime.

A. Machine Modeling

Consider the system in Fig. 2. It is assumed that a high-gain, servo-controlled motor capable of velocity control is used to produce displacement, x_m. This implies that motor friction is small compared to the rest of the system and that servo feedback is sufficient to produce any desired displacement. The motor is coupled to a mass, m, sliding with displacement x_l on a frictional surface. The coupling is modeled as a spring and dashpot in parallel. The spring represents the stiffness of the transmission while the dashpot models damping in the same. These elements can also be interpreted as PD feedback. This is the simplest machine model incorporating both friction and flexibility. The only nonlinear component of the system in Fig. 2 is friction which we assume can be linearized as discussed above.

Given the input, $\dot{x}_m = V_0$, the response of the system can be linearized about $\dot{x}_l = V_0$. The linearized system can be represented

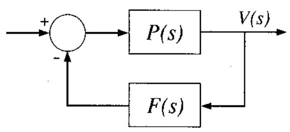


Fig. 3. Block diagram of sliding system model.

by the block diagram of Fig. 3 where $V(s)$ is the Laplace transform of the block's velocity perturbation about V_0 and

$$P(s) = \frac{s}{ms^2 + k_v s + k_p} \tag{11}$$

$$F(s) = f_v - \int_0^\infty g(t)e^{-st}\,dt = f_v - G(s). \tag{12}$$

Assume first that there is no frictional memory and $f = f(V(t))$. In this case,

$$\frac{\partial f}{\partial V}(V_0) = f_v = \frac{df_{ss}(V)}{dV} < 0 \tag{13}$$

Examination of the characteristic equation of this linearized system shows that for asymptotic stability, $(f_v + k_v) > 0$ and $k_p > 0$.

Example: Consider the friction model

$$F(V) = c_0 + c_1 V + c_2 e^{-(V/c_3)^2}. \tag{14}$$

In this case,

$$f_v = \frac{\partial f}{\partial V} = c_1 - \frac{2c_2 V}{c_3^2} e^{-(V/c_3)^2}. \tag{15}$$

If $f_v < 0$, a positive k_v is needed to stabilize the system. The hardest velocity to stabilize (requiring the maximum k_v) is found to be

$$V_{cr} = \frac{c_3}{\sqrt{2}}. \tag{16}$$

For asymptotic stability, $k_v > -f_v(V_{cr})$ and $k_p > 0$ where

$$f_v(V_{cr}) = c_1 - \frac{\sqrt{2}c_2}{c_3} e^{-1/2}. \tag{17}$$

To evaluate the stability of the system with a dynamic friction model, we introduce a theorem due to Rice and Ruina [11].

Theorem: Given the system for Fig. 3 described by (11) and (12) for which the function $g(t)$ is piecewise continuous for $t \in (0, \infty)$ and total damping is insufficient to stabilize the system, i.e.,

$$f_v + k_v < \int_0^\infty g(t)\,dt \tag{18}$$

then:

- The implicit equation

$$k_v + f_v = \int_0^\infty g(t)\cos(\omega_n t)\,dt \tag{19}$$

  has a nonzero, finite number of solutions, ω_n, with associated stiffnesses defined by

$$k(\omega_n) = m\omega_n^2 + \omega_n \int_0^\infty g(t)\sin(\omega_n t)\,dt. \tag{20}$$

- For $k_p > k_{cr} = \max_n \{k(\omega_n)\}$, the system is asymptotically stable where k_{cr} is the critical value of stiffness associated with undamped oscillations of frequency, $\omega_{cr} = \{\omega_n | k(\omega_n) = k_{cr}\}$.

Proof: See [11]. The theorem can be derived from the characteristic equation or from Nyquist criteria. Equations (19) and (20) can be obtained from the real and imaginary parts of the characteristic equation evaluated on the imaginary axis. The proof requires that $G(s) = \int_0^\infty g(t)e^{-st}\,dt$ meet the following conditions:

- For $\mathrm{Re}(s) \geq 0$, $G(s)$ is bounded and $\lim_{s\to\infty} G(s) = 0$.
- The zeros of the characteristic equation $1 + P(s)F(s)$ depend continuously on the stiffness parameter k_p.

The first set of conditions on $G(s)$ follows from (10) which indicates that the integral exists and converges uniformly. The condition that $g(t)$ be piecewise continuous is a sufficient, but not necessary, condition to guarantee the existence of a solution to (19) and thus to (20). The additional, optional constraint that $g(t) \geq 0$ can be added to ensure the uniqueness of ω_n in (19).

Example 1: For the state variable friction law of (3) and (4),

$$f_v = A/V_0 \tag{21}$$

$$g(t) = \frac{Be^{-V_0 t/L}}{L} \tag{22}$$

$$G(s) = \frac{B}{(s + V_0/L)L}. \tag{23}$$

Since $g(t) \geq 0$ and continuous, for a fixed velocity gain of k_v, the theorem gives the unique critical frequency and stiffness to be [11]

$$\omega_{cr} = \frac{V_0}{L}\sqrt{\frac{B}{A + k_v V_0} - 1} \tag{24}$$

$$k_{cr} = \frac{B - (A + k_v V_0)}{L}\left[\frac{mV_0^2}{(A + k_v V_0)L} + 1\right]. \tag{25}$$

Since it was assumed that velocity feedback was insufficient to stabilize the system, ω_{cr}, will be positive and real.

Example 2: For the time delay friction law of (6), define $k_f > 0$ as

$$\frac{df_{ss}(V_0)}{dV} = c_0 - \frac{2c_1 c_2 V_0}{(1 + c_2 V_0^2)^2} = -k_f < 0 \tag{26}$$

assuming the steady state friction-velocity curve is negatively sloped at V_0. The linearized friction law is described by

$$f_v = 0 \tag{27}$$

$$g(t) = k_f \delta(t - \alpha). \tag{28}$$

$$G(s) = k_f e^{-\alpha s}. \tag{29}$$

In this case, $g(t)$ is an impulse delayed by time α where $\delta(\cdot)$ is the unit impulse function. The conditions of the proof are not satisfied since, for $\mathrm{Re}(s) \geq 0$, $G(s)$ is not bounded and $\lim_{s\to\infty} G(s) \neq 0$. In addition, while $g(t) \geq 0$, it is not piecewise continuous. The solutions of (19) and (20) are given by

$$\omega_n = \frac{1}{\alpha}[\cos^{-1}(k_v/k_f) + 2n\pi], \quad n = 1, 2, 3, \cdots \tag{30}$$

$$k(\omega_n) = m\omega_n^2 + k_f \omega_n \sin(\alpha\omega_n). \tag{31}$$

Clearly there are an infinite number of ω_n and $k(\omega_n)$ which increase without bound for increasing n. Thus, while intervals of k_p yielding local stability may exist, there is no critical stiffness for the time-delay model. A complete stability analysis of the system is beyond the scope of this note. The interested reader is referred to [14] from which it is easy to show that, for certain choices of model parameters, the system alternates between exponential asymptotic stability and instability for large, increasing values of stiffness, k_p.

IV. Conclusions

The ability to achieve steady, low velocity motion can be important for machines with tasks involving fine positioning or force control. In practice, however, the highly nonlinear behavior of friction near zero velocity imposes a minimum stable velocity below which stick-slip occurs. This places a limit on position and force resolution. In addition, the overshoot associated with limit cycling can lead to task failure.

In this paper, the ramifications of history-dependent friction in the PD control of steady, low-velocity motion have been explored. While models which neglect frictional memory predict stable sliding only through the addition of sufficient damping, experiment and analysis indicate that steady motion can also be obtained by stiffening a system. Rice and Ruina's stability theorem describes an entire class of friction models which exhibit a critical value of system stiffness above which constant-velocity motion is asymptotically stable.

The state variable friction model of Ruina falls within the class of models exhibiting a critical stiffness. The model has recently been validated for several engineering materials and lubricants [6, 8]. Time-delay models, such as the one proposed by Hess and Soom [9], can predict intervals of stiffness which are stabilizing, but do not predict a critical stiffness. This suggests that their applicability may be limited to restricted regions of a system's parameter space.

In deriving our system model, k_v and k_p were interpreted as the inherent damping and stiffness of the system. Of course, they can also be interpreted as the feedback gains of the controller. In this light, the preceding analysis provides lower bounds on PD controller gains for stabilizing a system operating within the negatively-sloped region of the steady-state friction-velocity curve.

Acknowledgment

The author would like to thank Dr. James Rice of Harvard University for his valuable discussions of state variable friction models.

References

[1] B. Armstrong-Hélouvry, *Control of Machines with Friction.* Norwell, MA: Kluwer Academic Press, 1991.

[2] B. Armstrong-Hélouvry, "Stick-slip and control in low-speed motion," *IEEE Trans. Automat. Contr.*, vol. 38, no. 10, pp. 1483–1496, Oct. 1993.

[3] C. Canudas, K. Åstrom, and K. Braun, "Adaptive friction compensation in DC motor drives," *IEEE J. Robot. Automat.*, vol. RA-3, no. 6, pp. 681–85, Dec. 1987.

[4] C. Canudas D. Wit, P. Noel, A. Aubin, and B. Brogliato, "Adapative friction compensation in robot manipulators: Low velocities." *Int. J. Robot. Research*, vol. 10, no. 3, pp. 189–99, 1991.

[5] P. R. Dahl, "Measurement of solid friction parameters of ball bearings." in *Proc 6th Annual Symp. Incremental Motion Control Systems Devices*, Univ. Illinois, 1977, pp. 49–60.

[6] J. H. Dieterich, "Micro-mechanics of slip instabilities with rate- and state-dependent friction," *Eos, Trans. Am. Geophys. Union*, Fall meeting abstract vol., 1991, p. 324.

[7] P. Dupont, "The effect of friction on the forward dynamics problem," *Int. J. of Robotics Research*, vol. 12, no. 2, pp. 164–79, Apr. 1993.

[8] P. Dupont and E. Dunlap, "Friction modeling and control in boundary lubrication," in *Proc. 1993 Am. Control Conf.*, San Francisco, CA, June 1993, pp. 1910–1914.

[9] D. Hess and A. Soom, "Friction at a lubricated line contact operating at oscillating sliding velocities," *ASME J. Tribology*, vol. 112, pp. 147–52, Jan. 1990.

[10] E. Rabinowicz, "A study of the stick-slip process," in *Friction and Wear*, R. Davies (Ed.). New York: Elsevier, 1959.

[11] J. Rice and A. Ruina, "Stability of steady frictional slipping," *J. Applied Mechanics*, vol. 50, pp. 343–49, June 1983.

[12] A. Ruina, "Friction laws and instabilities: A quasistatic analysis of some dry frictional behavior," Ph.D. dissertation, Division of Engineering, Brown Univ., Providence, RI, 1980.

[13] J. Sampson, F. Morgan, D. Reed, and M. Muskat, "Friction behavior during the slip portion of the stick-slip process." *J. Applied Physics*, vol. 14, pp. 689–700, Dec. 1943.

[14] G. Stépán, *Retarded Dynamical Systems: Stability and Characteristic Functions.* Essex, UK: Longman Scientific and Technical, 1989.

[15] C. Walrath, "Adaptive bearing friction compensation based on recent knowledge of dynamic friction," *Automatica*, vol. 20, no. 6, pp. 717–27, 1984.

[16] S. Yang and M. Tomizuka, "Adaptive pulse width control for precise positioning under the influence of striction and coulomb friction," *ASME J. Dyn. Sys., Meas. Contr.*, vol. 110, pp. 221–227, Sept. 1988.

New Sufficient Conditions for Linear Feedback Closed-Loop Stackelberg Strategy of Descriptor Systems

Hua Xu and Koichi Mizukami

Abstract—**This paper is concerned with the derivation of linear feedback closed-loop Stackelberg (LFCLS) strategies for a class of continuous-time two-person nonzero-sum differential games characterized by linear descriptor systems and the quadratic cost functionals. Compared with existing results, new sufficient conditions for the existence of a LFCLS strategy are obtained under less restrictive conditions. Furthermore, from the results of this paper, we also arrive at some conclusions about the linear-quadratic closed-loop Nash game for continuous-time descriptor systems. These conclusions are distinguished from those of the linear-quadratic closed-loop Nash game for state-space systems.**

I. Introduction

The Stackelberg game problem for state-space systems is by now fairly completely worked out. Several papers exist which deal with the derivation of various closed-loop Stackelberg solutions of the continuous-time linear-quadratic game [2]–[6]. It is generally not possible to find a linear feedback closed-loop Stackelberg (LFCLS) strategy for such state-space systems.

In a recent paper [7], the linear-quadratic closed-loop Stackelberg game for continuous-time descriptor systems was investigated, and it was found that there exists an LFCLS strategy for the Stackelberg game of the descriptor system. This feature makes the Stackelberg game of the descriptor system different from the one of the state space system. As a basic assumption, however, it was required in [7] that the system be impulsively controllable with respect to the control of the leader. In other words, there exists a descriptor variable feedback control of the leader such that the closed-loop system has no dynamic modes at infinity if the control of the follower is zero.

In this paper, we shall show that the assumption given in [7] is conservative. We shall refine and improve the method developed in [7]. New sufficient conditions are obtained for the existence of a LFCLS strategy, and the techniques used are inherently different from those in [7]. The results obtained in this paper will extend

Manuscript received September 28, 1992; revised December 22, 1992, April 2, 1993, and May 17, 1993.

The authors are with the Faculty of Integrated Arts and Sciences, University of Hiroshima, 1-7-1, Kagamiyama, Higashi-Hiroshima, Japan 724.

IEEE Log Number 9400339.