The Effect of Coulomb Friction on the Existence and Uniqueness of the Forward Dynamics Problem

Pierre E. Dupont
Aerospace & Mechanical Engineering
Boston University
Boston, MA 02215

Abstract

For motion planning, it is important to have an accurate dynamic system model and, in many cases, friction is an important component of the model [1,2]. It is known, however, that when Coulomb friction is added to the rigid-body dynamic equations, the forward dynamic solution may not exist and if it exists, it is not necessarily unique. In this paper, we study the existence and uniqueness properties of the forward solution of these equations. We show that existence and uniqueness problems arise even for a single degree of freedom system and derive conditions under which such problems occur. A graphical method is presented which clearly shows the number of solutions associated with each value of input torque. A transmission element is used as an example of such a system.

1 Introduction

Beginning at the turn of the century, a number of researchers have shown that when the simple Coulomb friction model is combined with the rigid-body dynamic equations, there are cases when no solutions exist and also cases when multiple solutions occur. These cases have been treated separately in prior years at this conference [7,9]. As we will see, the conditions producing solution nonexistence and multiplicity are closely related.

The ability to include effects such as friction in the modeling and simulation of robotic systems is becoming more important especially for tasks such as tactile sensing and teleoperation. Thus there is the need to understand the nature of the existence and uniqueness problems and to know under what conditions these problems arise.

In the next section, we discuss the Coulomb friction model and prior work on the existence and uniqueness problem. In section 3, we present the forward dynamic equations and discuss their solution. In section 4, the scalar case is considered in detail and we present a necessary and sufficient condition for solution existence and uniqueness. Next, a screw transmission drive is discussed as an example of such a system. The paper concludes with a discussion of the results.

2 Background

The Coulomb friction force is directed so as to oppose relative motion and is proportional to the normal force of contact during motion. For unilateral constraints, the normal force F_n must be positive. Thus we can express Coulomb friction as

$$\begin{array}{l} F_n \geq 0 \\ v \neq 0 \Rightarrow |F_f| = \mu F_n, \; vF_f \leq 0 \\ v = 0 \Rightarrow |F_f| \leq \mu F_n \end{array} \tag{1}$$

where F_f is friction force, $\mu > 0$ is the coefficient of friction and v is the velocity of relative motion. These equations define what is commonly called the friction cone. During motion, the friction force must lie on the friction cone while during static contact, it may also lie inside the cone.

In actuality, friction behavior is more complicated than that of the Coulomb model. Friction can depend strongly on velocity [1], exhibit a nonlinear dependence on normal force and also exhibit transient behavior [3]. Nevertheless, Coulomb friction is an appropriate first-order model for many material combinations. It is therefore worthwhile to consider its effect on the existence and uniqueness problem.

This problem was first presented by Painlevé [8]. More recently, Lötstedt [6] has published a good derivation of the planar equations in terms of constraint forces and has provided conditions for solution consistency. He also presents an example of both solution nonexistence and multiplicity.

Two pertinent papers have appeared at this conference. Rajan et al considered friction in the context of a planar peg-in-hole problem and discussed the possibility of multiple solutions [9]. For the cases of a polygon in contact with one and two rigid walls, they mapped the type of motion associated with all possible force / torque pairs applied to the polygon. For contact with a single wall, they derive a condition (their equation 2.12) for which a range of input forces are consistent with three possible solutions – static contact, sliding contact and motion away from the wall.

The following year, Mason and Wang considered the case of a slender rod in contact with a single rigid wall [7]. They derived the condition (their equation 20) under which no solution consistent with Coulomb friction exists. Taking into account the differences in notation, Rajan's condition for multiple solutions is in fact the same as Mason's condition for nonexistence. This identical condition depends only on inertia, geometry and friction properties. Thus we make the interesting observation that, at least for the planar single contact case, the same system can possess either no consistent solutions or several. The number of solutions depends on the input force and torque. This observation can also be deduced from Lötstedt's paper [6].

For the case of no consistent solution, Mason and Wang concluded that under these conditions, the initially sliding rod stops suddenly and modeled it as an impact with zero approach velocity [7,11].

3 Forward Dynamic Equations

The rigid-body dynamic equations for a mechanical system such as an open-kinematic-chain robot are of the form

$$\tau = A(q)\ddot{q} + b(q,\dot{q}) + f(q,\dot{q},\ddot{q}). \tag{2}$$

The n-vectors of generalized coordinates (such as joint positions) and associated input forces or torques are q and τ, respectively where n is the number of degrees of freedom of the system. The configuration-dependent inertia matrix is represented by $A \in \mathbf{R}^{n\times n}$. It is both symmetric and positive definite. The vector $b \in \mathbf{R}^n$ consists of centrifugal, Coriolis and gravity terms. The vector $f \in \mathbf{R}^n$ includes all friction terms and is a function of the generalized coordinates and their first and second derivatives.

The *forward dynamics problem* is to solve for the joint positions, velocities and accelerations given the input torques or forces and the initial conditions. This is the problem to be solved for simulation. At each time step, the known joint torques, positions and velocities are used to compute the joint accelerations. In the absence of friction, this involves solving a set of linear algebraic equations for the accelerations. Using the values of acceleration and velocity, numerical integration yields the velocity and position at the next time step.

Friction can arise due to relative motion between the rigid bodies making up the mechanism or due to contact between one or more of these bodies and the environment. We will refer to these two types as *internal* and *external* friction, respectively. Internal friction is due to such elements as the transmissions and bearings. External friction acts at contacts between the robot and its environment. It is important in grasping and assembly operations.

In the case of external friction, (2) must be modified as follows.

$$\tau = A(q)\ddot{q} + b(q,\dot{q}) + \sum_i J_i^T(q)F_{n_i} + \sum_i J_i^T(q)f(q,\dot{q},\ddot{q}). \tag{3}$$

This equation assumes multiple contact points along the links. Here, F_{n_i} is the normal force vector at contact point i and J_i is the Jacobian relating infinitesimal joint and contact point displacements. Similarly, the friction vector is premultiplied by the transpose of the Jacobian matrix.

If the generalized coordinate directions can be chosen so as to coincide with the directions of the external normal and friction forces then the Jacobian reduces to the identity matrix and (3) reduces to (2). This is true of the problems discussed by Rajan et al [9] and by Mason and Wang [7]. In this paper, we will discuss only those cases which satisfy (2).

3.1 Implicitness of the Forward Solution

In addition to existence and uniqueness issues, the inclusion of load-dependent Coulomb or static friction in the robot dynamic equations typically renders them implicit in the joint accelerations. Thus, even if it is known that a unique solution for the accelerations exists, it must, in general, be obtained using an iterative root-finding process at each step of the integration [2,5].

The cause of the implicitness is the dependence of Coulomb friction on the magnitude of the normal force. The normal force itself is a function of the resultant force and moment at the friction contact. Expressed in a local coordinate frame, the components of the resultant force and moment can be formulated in terms of the joint positions, velocities and accelerations [2].

If the direction of the normal force happens to be constant in the local frame, the normal force can be expressed as a function in which the net force and moment components appear linearly. This is true, for example, of friction in transmissions and translational joints. Since the sign of the normal force at a bilateral constraint can change, its absolute value must be used to obtain its magnitude [2].

When the direction of the normal force is not constant in a local joint coordinate frame, the magnitude of the normal force will involve the square root of sums of squares of net force and moment components [5]. This would be true of radially-loaded revolute bearings. Consequently, Coulomb friction can involve either the absolute value or square root of sums of squares of acceleration-dependent terms. Therefore, it is necessary to solve iteratively for joint accelerations at each time step of a simulation.

3.2 Internal Friction

Consider the case of a single source of internal friction associated with each degree of freedom and assume that the direction of the normal forces are fixed in their respective local coordinate frames. Typically, internal friction arises from bilateral constraints. In this case, the friction vector, f, becomes

$$f(q,\dot{q},\ddot{q}) = M|C(q)\ddot{q} + d(q,\dot{q})|. \tag{4}$$

Here $M \in \mathbf{R}^{n\times n}$ is a diagonal matrix with diagonal elements $\mu_i sgn(\dot{q_i})$ and $\mu_i > 0$ are the coefficients of friction. $C \in \mathbf{R}^{n\times n}$ takes the form of an inertia matrix and $d \in \mathbf{R}^n$. The expression in absolute values is the normal force.

In this paper, we will study friction forces of the form given by (4). The external friction problems of [7] and [9] can be posed in this form.

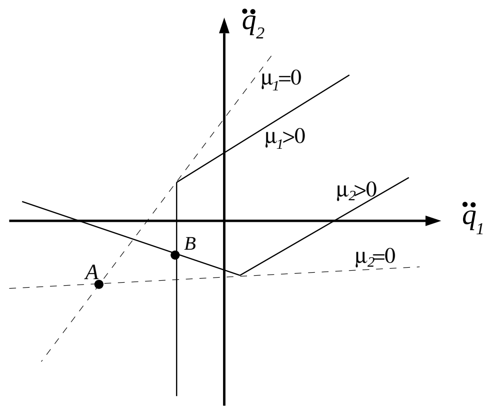

Figure 1: Constraint Equations in Acceleration Space. The frictionless solution corresponds to the intersection of the two dashed lines, point A. For $\mu_i > 0$, the V-shaped constraint equations intersect at B yielding a unique solution. By reorienting the V's and changing their included angles, cases of zero and multiple intersection points can be obtained.

The system of equations is given by

$$A\ddot{q} + b + Diag[\mu_i \text{sgn}(\dot{q}_i)]|C\ddot{q} + d| = \tau \tag{5}$$

in which A, C, b, d, $\text{sgn}(\dot{x}_i)$ and τ are known constants and $|\cdot|$ denotes vector absolute value, not norm. The solution set consists of all intersection points of the constraint equations.

For $\mu_i = 0$, the equations simplify to

$$A\ddot{q} = b \tag{6}$$

Each constraint equation describes a hyperplane in the space of joint accelerations and there is always a unique solution since A corresponds to the inertia matrix which is invertible.

For $\mu_i > 0$ and n joints, the constraints are V-shaped, $(n-1)$-dimensional half-hyperplanes. Their intersection may be empty or may consist of multiple points in acceleration space. All solutions found are consistent with the equations and represent the dissipation of energy by friction. Figure 1 provides a two degree of freedom example.

4 Scalar Case

The planar problem most often used to illustrate existence and uniqueness issues is that of a slender rod in contact with a single immobile wall. The rod has two degrees of freedom while in contact with the wall and three otherwise. In this section, it is shown that the forward dynamics of a single degree of freedom system exhibit the same existence and uniqueness problems as the multiple degree of freedom rod.

The scalar form of (5) is given by

$$a\ddot{q} + b + \mu|c\ddot{q} + d|\text{sgn}(\dot{q}) = \tau \tag{7}$$

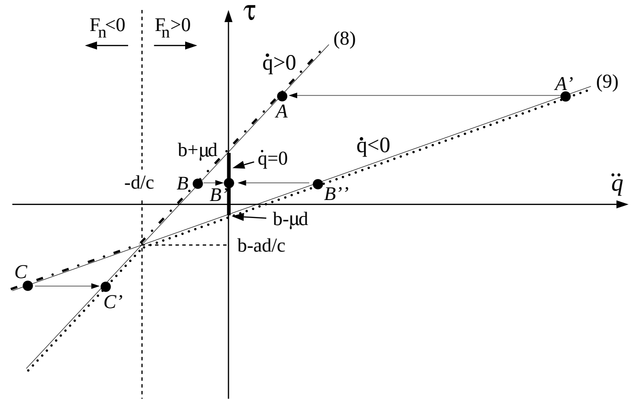

Figure 2: Scalar Case of τ Versus $\ddot{q}$ for $\mu < a/c$ and $d > 0$. The upper, dot-dashed V corresponds to $\dot{q} > 0$. The lower, dotted V corresponds to $\dot{q} < 0$. The dark line segment on the τ-axis is the static region where $\dot{q} = 0$. The horizontal arrows indicate discontinuities in acceleration which occur when $\dot{q} = 0$.

Note that the single normal force, F_n, is given by $F_n = c\ddot{q} + d$. We have two equations:

$$(a + \mu c)\ddot{q} + (b + \mu d) = \tau, \quad \text{sgn}(\dot{q}) = \text{sgn}(F_n) \tag{8}$$

$$(a - \mu c)\ddot{q} + (b - \mu d) = \tau, \quad \text{sgn}(\dot{q}) \neq \text{sgn}(F_n) \tag{9}$$

These equations are intersecting lines in the space formed by τ and $\ddot{q}$. The case of $\mu < a/c$ and $d > 0$ is shown in Figure 2. Note that a and c, representing inertias, must be positive.

Since these equations represent bilateral constraints, normal force can be of either sign and $F_n = 0$ along the vertical line through $\ddot{q} = -d/c$. Thus for $\dot{q} > 0$, the system lies on the upper V and for $\dot{q} < 0$, on the lower V. If the system is initially static, it lies on the τ axis between the two lines. From this static region, if the torque is increased above $b + \mu d$, motion with $\dot{q} > 0$ ensues. Similarly, if the the torque is decreased below $b - \mu d$, motion with $\dot{q} < 0$ ensues. At $b \pm \mu d$, the friction force lies on the friction cone associated with $\ddot{q} = 0$.

Consider the behavior of various points on the graph. At A, both the velocity and acceleration are positive. At B, acceleration is negative while the velocity is positive, but decreasing. Since point B is within the static band on the τ-axis, the system will stick when the velocity reaches zero. As shown by the arrow, the system jumps to B' and the acceleration discontinuously jumps to zero. Point C, however, is outside the static region. When $\dot{q} = 0$ is reached at C, the system jumps to C' on the negative-velocity V. For $\dot{q} < 0$ initially, points B'' and A' lead to sticking and velocity reversal, respectively.

System behavior in the case of unilateral constraints can also be determined from this graph. In this case, $F_n < 0$ would correspond to a cessation of contact at the friction interface. At points to the left of the vertical line $\ddot{q} = -d/c$,

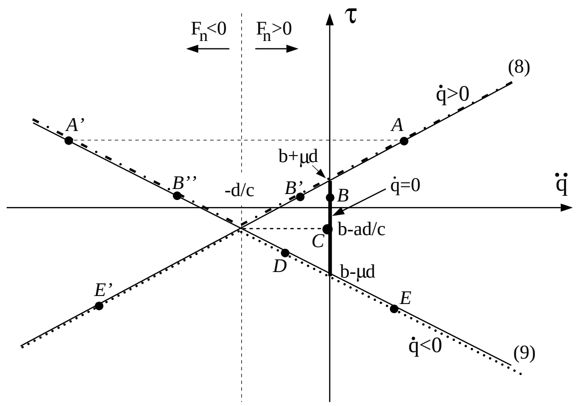

Figure 3: Scalar Case of τ Versus $\ddot{q}$ for $\mu > a/c$. In this case, there can be multiple solutions or none.

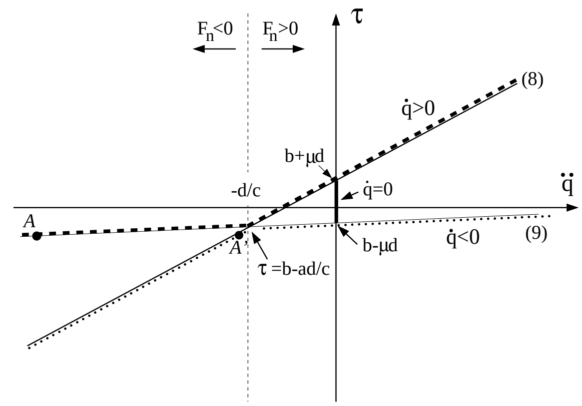

Figure 4: Near-critical Case of τ Versus $\ddot{q}$. Here, the line of (9) is nearly horizontal. For $\dot{q} > 0$, applying $\tau < (b - ad/c)$ moves the system to a point such as A producing a very large negative acceleration.

the system flies apart. In the slender rod example, contact between the rod and wall would cease.

Figure 2, together with the current values of velocity and torque, provide a unique value of acceleration for all but two points. For example, consider that the three values of $\ddot{q}$ at B, B' and B'' are associated with a single value of torque. These solutions, however, correspond to $\dot{q} > 0$, $\dot{q} = 0$ and $\dot{q} < 0$, respectively. The two ambiguous points occur for $\tau = b \pm \mu d$. Note that a static friction coefficient, $\mu_s > \mu$, would resolve this ambiguity.

Let us now consider the conditions under which solution existence and uniqueness fail for most values of input torque.

4.1 Solution Existence and Multiplicity

Consider the graph of (8) and (9) in Figure 3. Here, $\mu > a/c$ and so (9) has a negative slope. This corresponds to a negative effective inertia.

Once again, the upper V is associated with positive velocities and the lower with negative velocities. Starting from rest, the input torque must leave the static region to initiate motion. For example, if the torque is increased above $b + \mu d$, motion in the positive direction will ensue. Now, however, there are two possible solutions given, for example, by A and A'. If the torque is decreased to the value associated with B, there are three possible values of acceleration given by B, B' and B''. B corresponds to the static case. B' and B'' both correspond to $\dot{q} > 0$, but with different values of acceleration, $\ddot{q}$.

If the torque is decreased slightly below $b - ad/c$, the only valid solution is the static case C, but as with the previous figure, the system would normally only jump like this when the velocity reached zero. Thus, there appear to be no solutions provided by the graph for $\dot{q} > 0$ and $\tau < (b - ad/c)$. We can make the same arguments for negative velocities. According to the value of input torque and velocity, we can have from zero to three feasible solutions.

4.1.1 Existence

To understand what actually happens in the case when $\dot{q} > 0$ and τ is decreased below $(b - ad/c)$, consider the near-critical case shown in Figure 4 when the line (9) is almost horizontal. This occurs when $\mu = a/c + \epsilon$ for small ϵ. Assume that initially $\dot{q} > 0$ and $\tau > (b - ad/c)$. A slight decrease in τ below $(b - ad/c)$ generates an extremely large negative acceleration. The velocity will rapidly fall to zero and the system will jump to A' on the negative velocity V. As the value $\mu = a/c$ is approached, deceleration time approaches zero as does the sliding distance.

If, as in Figure 3, $\mu > a/c$, the applied force or torque, τ, is insufficient to shear the bonds at the friction interface. Thus, according to Coulomb's model, no sliding can occur and the velocity must jump discontinuously to zero. This agrees with Wang and Mason's conclusion that an impulse analysis is in order [7].

4.1.2 Multiplicity

The analyses of Rajan et al, [9], and Lötstedt, [6], produce cases of three possible solutions of the type given by B, B' and B'' in Figure 3. They do not, however, provide a method of selecting the "correct" solution.

The rigid-body analysis provides only a partial answer. Since the static case can only be escaped by applying a torque outside of the static region bounded by $b \pm \mu d$, B is the only possible solution if $\dot{q}$ is initially zero.

Solution pairs such as (A, A') and (B', B'') all correspond to $\dot{q} > 0$, however the points of each pair do differ by the sign of normal force. If normal force could be formulated as an explicit function of system state, a unique solution would be known. This approach is taken in [4] by considering non-rigid bodies.

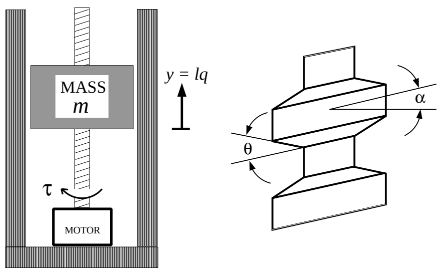

Figure 5: Screw-driven Mass. The motor applies torque τ to the screw. The mass, m, is attached to the nut. Its displacement, y, is related to the screw displacement, q, by the screw lead, l. The helix angle, α and thread angle, θ, are also shown.

4.1.3 Necessary and Sufficient Condition

In summary, a necessary and sufficient condition for solution existence and uniqueness for arbitrary values of input torque is given by

$$\mu < a/c \tag{10}$$

When $\mu \geq a/c$, solution existence and uniqueness problems arise only for $\dot{q} \neq 0$ and depend on the applied torque history.

5 Example: Screw Drive

Let us now elucidate the discussion of the preceding section by means of an example. Screws are sometimes used as transmission elements in robots. See, for example, [2].

Consider the case of a screw moving a mass m with gravity acting downward as shown in Figure 5. The following definitions will be used.

q = screw displacement	y = mass displacement
l = screw lead	$y = lq$
α = screw helix angle	θ = screw thread angle
I = screw inertia	$\tan\rho = \mu/\cos\theta$

While the screw imparts a vertical force, F_m, to the mass, we can express this as a torque, τ_l, by multiplying it by the screw lead, l.

$$\tau_l = lF_m = ml^2\ddot{q} + mgl \tag{11}$$

It can be shown that the normal force on the screw threads is of the same sign as τ_l,

$$\mathrm{sgn}(F_n) = sgn(\tau_l) \tag{12}$$

Using the expressions for screw efficiency, η, found in standard mechanical design texts, such as [10], the dynamic equations are

$$(I + ml^2/\eta_i)\ddot{q} + mgl/\eta_i = \tau$$

$$\eta_i = \begin{cases} \eta_1 = \frac{\tan\alpha}{\tan(\alpha+\rho)}, & \mathrm{sgn}(\tau_l) = \mathrm{sgn}(\dot{q}) \\ \eta_2 = \frac{\tan\alpha}{\tan(\alpha-\rho)}, & \mathrm{sgn}(\tau_l) \neq \mathrm{sgn}(\dot{q}) \end{cases} \tag{13}$$

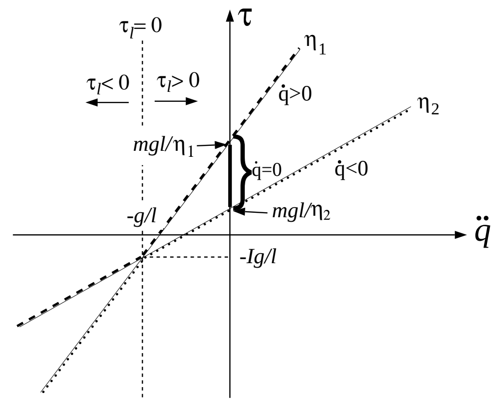

Figure 6: Graph of τ Versus $\ddot{q}$ for an Overhauling Screw. Since $\alpha > \rho$, a positive torque is required to maintain $\dot{q} = 0$.

These are two linear equations in $(\ddot{q}, \tau)$ which are of the same form as (8) and (9).

5.1 Necessary and Sufficient Condition

A necessary and sufficient condition for solution existence and uniqueness is to require that the effective inertias, which are also the slopes of the lines, be positive. This condition is

$$(I + ml^2/\eta_i) > 0 \tag{14}$$

In this case, the graph of the equations will be similar to Figure 2.

This condition is always met if the following inequality is satisfied.

$$\alpha > \rho \tag{15}$$

A screw of this type is called backdrivable or overhauling due to its low friction. Unless a sufficiently-positive torque is applied to the screw under static conditions, the mass will descend due to gravity. This can be seen from its graph, Figure 6, in which the static region lies entirely on the positive half of the τ-axis since $mgl/\eta_2 > 0$.

Equation (15) is even more restrictive than (14) and thus is a sufficient, but not necessary condition. It is, however, an important case because almost all screws used as transmissions in robots and mechanisms are of this type.

5.2 Screw Behavior

As in the general scalar case, solution existence and uniqueness problems arise when one line of the graph is horizontal or negatively sloped. For this to be true, $\alpha < \rho$ and so existence and uniqueness is only an issue with nonoverhauling screws. In addition, the product ml^2 must be large compared to the rotational screw inertia. This can be compared to the slender rod problem for which existence and uniqueness problems can only arise when the rotational inertia is large compared to the mass.

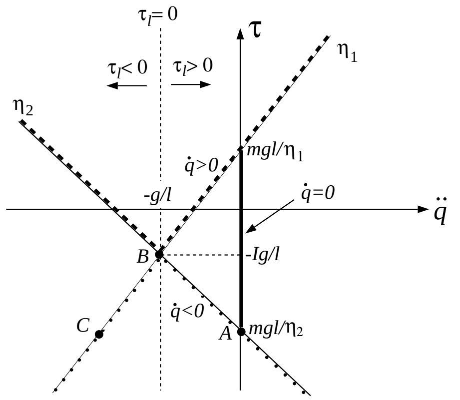

Figure 7: Graph of τ Versus $\ddot{q}$ for a Screw with $(I + ml^2/\eta_2) < 0$.

The graph for this case appears in Figure 7. We can gain additional insight into its V-shape by considering the case of lowering the mass starting from rest. With $\ddot{q} = 0$, the normal force on the screw threads is entirely due to gravity. Due to the high friction coefficient, a negative torque, $\tau < mgl/\eta_2$, must be applied to overcome friction.

Following the graph from A to B, the negative acceleration of the screw decreases the gravity-load-induced normal force. As a result, the friction force decreases and a smaller magnitude of torque is needed to overcome friction.

At point B, $(\ddot{q}, \tau) = (-g/l, -Ig/l)$. The normal force between the screw and nut threads is zero. The mass is in free fall. Now consider moving from B towards C. The screw is pushing down on the nut. Larger negative accelerations generate larger normal forces and thus larger negative torques are needed to overcome friction.

6 Conclusion

We have shown that the forward dynamic equations for a single degree of freedom system can exhibit existence and uniqueness problems. In fact, the same system parameter values can produce cases of both no solution and multiple solutions depending on the value of input torque or force.

We derived a necessary and sufficient condition for solution existence and uniqueness. Coefficients of friction which do not satisfy this condition are quite large compared to typical magnitudes of internal friction. This is a significant result because it indicates that in many important cases, existence and uniqueness of the forward dynamics is not a problem.

Coefficients of this magnitude, however, might easily be encountered in external friction applications involving static or sliding contact with the environment. In these cases, it is necessary to identify the actual forward dynamic solution associated with a value of input torque. Toward this end, we have indicated when the static solution applies. The remaining ambiguity between multiple dynamic solutions arises because normal force cannot be expressed as an explicit function of system state with a rigid-body model. This ambiguity can be resolved by considering systems of finite stiffness [4]. It remains to be seen if higher dimensional systems yield even more interesting behavior.

Acknowledgement

I wish to thank Professor Guido Sandri who spent many hours discussing this topic with me.

7 References

1. B. Armstrong-Helouvry, *Control of Machines with Friction.* Kluwer Academic Press, Norwell, MA, 1991.
2. P. Dupont, "Friction Modeling in Dynamic Robot Simulation." *Proceedings 1990 IEEE Int. Robotics and Automation Conf.*, Cincinnati, OH, May 1990.
3. P. Dupont, "Avoiding Stick-Slip in Position and Force Control through Feedback," *Proceedings 1991 IEEE International Conference on Robotics and Automation*, Sacramento, CA, pp. 1470-1475, April 1991.
4. P. Dupont, "The Use of Compliance to Resolve the Existence and Uniqueness of the Forward Dynamics Solution with Coulomb Friction," *Proceedings CSME Forum 1992*, Montreal, PQ, Canada, June 1992.
5. A. Gogoussis and M. Donath, "Coulomb Friction Effects on the Dynamics of Bearings and Transmissions in Precision Robot Mechanisms." *Proceedings 1988 IEEE Int. Robotics and Automation Conf.*, Philadelphia, PA, pp. 1440-1446, 1988.
6. P. Lötstedt, "Coulomb Friction in Two-dimensional Rigid Body Systems." *Zeitschrift für Angewandte Mathematik un Mechanik*, 61, pp. 605-615.
7. M.T. Mason and Y. Wang, "On the Inconsistency of Rigid-body Frictional Planar Mechanics." *Proceedings 1988 IEEE Int. Robotics and Automation Conf.*, Philadelphia, PA, April, 1988.
8. P. Painlevé, "Sur les Lois du Frottement de Glissement." *Comptes Rendus de l'Académie des Sciences*, 121, pp. 112-115, 1895.
9. V.T. Rajan, R. Burridge and J.T. Schwartz, "Dynamics of a Rigid Body in Frictional Contact with Rigid Walls: Motion in Two Dimensions." *Proceedings 1987 IEEE Int. Robotics and Automation Conf.*, Raleigh, North Carolina, 1987.
10. M.F. Spotts, *Design of Machine Elements.* 6th Edition, Prentice Hall, New Jersey, 1985.
11. Y. Wang and M.T. Mason, "Two Dimensional Rigid Body Collisions with Friction." *Journal of Applied Mechanics, Trans. of ASME*, to appear, 1991.