To appear in *Journal of the Acoustical Society of America*, Vol. 112, No. 2, August 2005.

Realization of Mechanical Systems from Second-order Models

Wenyuan Chen* and Pierre E. Dupont†
Department of Aerospace and Mechanical Engineering,
College of Engineering, Boston University, Boston, Massachusetts, 02215
(Dated: May 23, 2005)

Congruent coordinate transformations are used to convert second-order models to a form in which the mass, damping and stiffness matrices can be interpreted as a passive mechanical system. For those systems which can be constructed from interconnected mass, stiffness and damping elements, it is shown that the input-output preserving transformations can be parameterized by an orthogonal matrix whose dimension corresponds to the number of internal masses – those masses at which an input is not applied nor an output measured. Only a subset of these transformations result in mechanically realizable models. For models with a small number of internal masses, complete discrete mapping of the transformation space is possible permitting enumeration of all mechanically realizable models sharing the original model's input-output behavior. When the number of internal masses is large, a nonlinear search of transformation space can be employed to identify mechanically realizable models. Applications include scale model vibration testing of complicated structures and the design of electro-mechanical filters.

PACS numbers: 43.40.At, 43.40.Sk

I. INTRODUCTION

The mechanical realization problem is the conversion of a passive input-output dynamic model to a form that is recognizable as an interconnected system of mechanical components. Applications of mechanical realization arise in those situations for which it is desirable to fabricate a mechanical system possessing specified input-output behavior.

An important example is the scale model testing of complicated structures in naval and aircraft design. While the major structural elements can be easily scaled and fabricated, scale models of other components, such as electronic equipment and machinery, are not easily manufactured. In these cases, the most efficient solution can be to model the input-output behavior of the equipment where it attaches to the major structure and to build a simple structure which is dynamically equivalent. Similarly, in the design of electro-mechanical filters[1], the desired input-output behavior is specified and its mechanical realization is sought.

The mechanical realization problem starts with the specification of a dynamic model describing input-output behavior. In the case of scale modeling, this model may be obtained through finite element analysis or estimation from experimental data. In filter design, the model will depend on the purpose of the filter. While both time-domain and frequency domain model descriptions are possible, this paper examines the realization problem for time-domain models specified in the second-order form,

$$\begin{aligned} M\ddot{q} + C\dot{q} + Kq &= Fu \\ y &= H_d q + H_v \dot{q} + H_a \ddot{q}. \end{aligned} \tag{1}$$

The $n \times 1$ vector q is the set of displacement coordinates, the $m \times 1$ vector u is the input vector, which is often an external force vector, and the $p \times 1$ vector y is the output vector. The mass matrix is $M = M^T > 0$, the damping matrix is $C = C^T \geq 0$ and the stiffness matrix is $K = K^T \geq 0$. F is the $n \times m$ input influence matrix, which is determined by the location of the input forces or torques. H_d, H_v and H_a are the output influence matrices of displacement, velocity and acceleration, respectively. In many circumstances, only accelerations need be considered as outputs and so $H_d = H_v = 0$, while $H_a \neq 0$. This is the case considered in this paper.

The mechanical realization problem for undamped or proportionally damped systems in the form of (1) has been widely studied. For these systems, the mass matrix can be reduced to diagonal form while the damping and stiffness matrices can be converted to either tridiagonal or border diagonal form. The former consists of a realization in which the masses are connected in series while, in the latter, they are connected in parallel. For example, a serial model can be obtained by Falk's algorithm using a congruent transformation computed from the given mass and stiffness matrices.[2, 3] Parallel realizations can be obtained using the normal mode theory of O'Hara and Cunniff.[4] Their results were generalized to a mechanical system undergoing three-dimensional vibration by Pierce[5].

The existence of structure-preserving transformations which result in diagonal mass, damping and stiffness matrices has been demonstrated for most real second order systems [6, 7]. While this form is amenable to numerical computation of input-output response by superposition,

*Currently at Servo Dynamics Corporation, 21541 Nordhoff Street, Chatsworth, California, 91311; Electronic address: wychen@alum.bu.edu
†Electronic address: pierre@bu.edu

it is not appropriate for mechanical realization which requires any superposition of responses to be performed mechanically.

A related body of work addresses inverse eigenvalue and inverse vibration problems. [8–11] The former is concerned with constructing a matrix with specified eigenvalues and so applies to the realization of mass normalized systems. The inverse vibration problem involves the reconstruction of mass and stiffness matrices from prescribed frequency response data, such as resonance frequencies. This approach can be extended to include proportional damping.

The question of whether or not an arbitrary $\{M, C, K, F, H\}$ corresponding to a passive system can be transformed to mechanically realizable form has not been addressed in the literature. It remains an open question, although one might anticipate that a result similar to the positive realness requirement of electrical network synthesis [12] also holds for mechanical systems. Furthermore, a recipe for transforming a system to mechanically realizable form is unknown. As a result, the realizability problem must be solved numerically using optimization algorithms.

The contribution of this paper is to characterize the set of transformations by which a class of models with viscous, but nonproportional damping can be converted to mechanically realizable form. The approach taken is to parameterize the set of transformations relating all input-output equivalent models which could result in a mechanically realizable form. Using this parameterization, mechanical realizations can be found by mapping or selectively searching the set of transformations. They can also guide future efforts seeking closed form solutions. These topics and examples are presented in the following sections.

II. STRUCTURE OF MECHANICALLY REALIZABLE SECOND ORDER MODELS

Motivated by the application of scale modeling equipment and machinery, this paper considers a specific subset of mechanically realizable systems consisting only of interconnected mass, stiffness and damping elements. Other types of elements, such as transmissions, are precluded. It is also assumed that there are no isolated masses in the system and that the system is statically stable, i.e., each mass is connected to the rest of the realization by at least one spring. Furthermore, the models are constrained to include a rigid body mode, i.e., they cannot employ skyhook connections comprised of springs and dashpots attached to a fixed ground.

For the intended applications, model simplicity drives the choice of mechanical elements while ease of implementation precludes the use of skyhook attachments. The results presented here can be adapted to permit additional model elements or to eliminate the rigid body mode. Both of these cases are less restrictive than the one considered since, for the former, the solution space is enlarged and, for the latter, the number of constraints is reduced.

Finally, only realizations corresponding to diagonal mass matrices are considered here. Block diagonal mass matrices involving, e.g., coupling between linear and rotational coordinates, may arise in practical applications, but are beyond the scope of this paper.

A. Realizable Stiffness and Damping Matrices

In addition to enforcing diagonality of the mass matrix, the conditions above also constrain the form of the stiffness and damping matrices. The simple mechanical system of Figure 1 is used to illustrate these properties, which are well known. Since these requirements are the same for both types of matrices, a realizable stiffness matrix is used to demonstrate them. The mass and stiffness matrices are expressed as follows

$$M = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 + k_4 & -k_1 & -k_2 & -k_4 \\ -k_1 & k_1 + k_3 & 0 & -k_3 \\ -k_2 & 0 & k_2 & 0 \\ -k_4 & -k_3 & 0 & k_3 + k_4 \end{bmatrix} \quad (2)$$

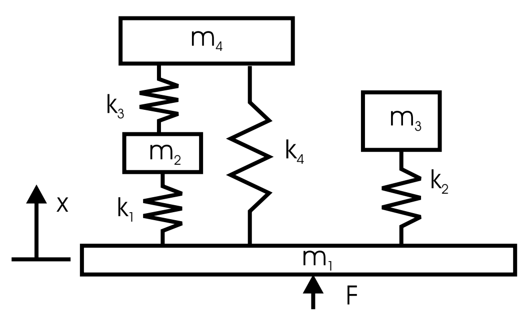

FIG. 1: Simple Mechanical Model

The stiffness matrix can be decomposed into the following form[13]

$$K = C_K K_D C_K^T \quad (3)$$

where the connectivity matrix C_K and nonnegative diagonal matrix K_D are given by

$$C_K = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$K_D = \begin{bmatrix} 0 & & & & \\ & k_1 & & & \\ & & k_2 & & \\ & & & k_3 & \\ & & & & k_4 \end{bmatrix} \tag{4}$$

The connectivity matrix C_K encodes the interconnection of masses by springs. The first column of C_K is chosen arbitrarily to represent the rigid body mode of the system in Figure 1, corresponding to the zero element of K_D. The other columns of C_K indicate connections between pairs of masses. For example, the third column represents the connection between m_1 and m_3 by stiffness k_2 in K_D. The opposite signs on the nonzero elements of these columns $(+1, -1)$ together with $K_{D_{11}} = 0$ ensures that K will have a nullspace vector $\left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]^T$ corresponding to a rigid-body mode. For a mechanical system with n masses and n_k springs, C_K is an $n \times (n_k + 1)$ matrix and K_D is an $(n_k+1) \times (n_k+1)$ diagonal matrix.

Similarly, a mechanically realizable damping matrix C can be decomposed as $C = C_C C_D C_C^T$, where C_C is a connectivity matrix and C_D is a diagonal matrix with nonnegative diagonal elements. It should be noted that while C_C is not necessarily equal to C_K, both will share the nullspace vector $\left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]^T$ ensuring the prohibition against skyhook springs and dashpots.

It can be summarized that the mechanically realizable mass, damping and stiffness matrices must satisfy the following realization conditions:

$$\begin{aligned} M &= diag(\left[\begin{array}{cccc} m_1 & m_2 & \cdots & m_n \end{array}\right]), m_i > 0 \\ C &= C^T, C_{ii} \geq 0, C_{ij} \leq 0, C \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]^T = 0 \\ K &= K^T, K_{ii} > 0, K_{ij} \leq 0, K \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right]^T = 0 \end{aligned} \tag{5}$$

where $i, j = 1, 2, \cdots, n$ and $i \neq j$.

B. Realizable Input and Output Influence Matrices

The input and output influence matrices can be categorized in terms of both the number of inputs and outputs as well as their relative locations. In the case of single-input, single-output (SISO) systems, the influence matrices are vectors while for multi-input, multi-output (MIMO) systems they are matrices. If the inputs and outputs are collocated then the system can be further classified as a driving-point realization while those systems with noncollocated inputs and outputs are termed transfer realizations.

Without loss of generality, it is assumed that the desired input and output influence vectors or matrices are given by

1. SISO Driving-point Accelerance

$$F = H^T = e_1 \tag{6}$$

2. SISO Transfer Accelerance

$$\begin{aligned} F_f &= e_1 \\ H_f^T &= e_2 \end{aligned} \tag{7}$$

3. MIMO Driving-point Accelerance

$$F_f = H_f^T = \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array}\right] \tag{8}$$

4. MIMO Transfer Accelerance

$$\begin{aligned} F_f &= \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array}\right] \\ H_f^T &= \left[\begin{array}{cccc} e_{m+1} & e_{m+2} & \cdots & e_{m+p} \end{array}\right] \end{aligned} \tag{9}$$

5. MIMO Driving-point and Transfer Accelerance

$$\begin{aligned} F_f &= \left[\begin{array}{cccc} e_1 & e_2 & \cdots & e_m \end{array}\right] \\ H_f^T &= \left[\begin{array}{ccccccccc} e_1 & e_2 & \cdots & e_r & e_{m+1} & e_{m+2} & \cdots & e_{m+(p-r)} \end{array}\right] \end{aligned} \tag{10}$$

Here, e_i is an element of the standard basis for $\Re^n$, which has a 1 at the i'th component and 0's elsewhere. The excitation forces are applied at a set of m coordinates and the accelerations are measured at a set of p coordinates. Both sets share r common coordinates.

III. TRANSFORMATIONS RELATING REALIZATIONS

Following the form of (1), an initial second order model describing the accelerance of a mechanical system is given by

$$\begin{aligned} M_0\ddot{x} + C_0\dot{x} + K_0 x &= F_0 u \\ y &= H_0\ddot{x}. \end{aligned} \tag{11}$$

The goal of the paper is to convert this initial model to one possessing the same input-output dynamic behavior, but which also satisfies the mechanical realizability conditions defined in the previous section. Congruent coordinate transformations can be seen to maintain the input-output behavior of the model while also preserving the symmetry of the mass, damping and stiffness matrices. Consider the coordinate transformation

$$x = Tq \tag{12}$$

where T is a nonsingular matrix. A congruence transformation converts the initial model (11) to the following form.

$$
\begin{aligned}
M_f\ddot{q} + C_f\dot{q} + K_f q &= F_f u \\
y &= H_f\ddot{q}
\end{aligned}
\tag{13}
$$

Here M_f, C_f, K_f, F_f and H_f are, respectively, the final mass, damping and stiffness matrices, and the input and output influence matrices. They are defined as

$$
\begin{aligned}
M_f &= T^T M_0 T \\
C_f &= T^T C_0 T \\
K_f &= T^T K_0 T \\
F_f &= T^T F_0 \\
H_f &= H_0 T
\end{aligned}
\tag{14}
$$

Thus, the set of invertible matrices $T \in \Re^{n\times n}$ describes the family of all second-order models satisfying input-output equivalence with (11) while preserving mass, damping and stiffness matrix symmetry. Only a subset of matrices T may result in a mechanically realizable model in which the final mass, damping and stiffness matrices satisfy (5) and the final input and output influence matrices satisfy one of (6)-(10).

While necessary and sufficient conditions for an initial model to be transformable to mechanically realizable form are not available, the following is a necessary condition for an initial model to possess a rigid body mode:

$$
C_0 v_0 = K_0 v_0 = 0 \tag{15}
$$

This equation states that the initial damping and stiffness matrices must share the same nullspace vector, v_0. This follows from $C\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T = 0$ and $K\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T = 0$ in (5) and the fact that congruence transformations preserve the signature of a matrix.

A. Decomposition of the Transformation

The coordinate transformation in (12) can be decomposed into a product of three components as follows.

$$
T = M_0^{-1/2} R M_f^{1/2} \tag{16}
$$

The first component, the inverse square root of the initial mass matrix, is used to mass normalize the initial second order model (11). The second component R is an orthogonal matrix, which preserves mass normalization. To obtain mechanically realizable form, it must perform two tasks. First, it should convert the input and output influence matrices to one of the desired forms (6)-(10). Secondly, from (5), it must ensure that all off-diagonal components of the damping and stiffness matrices are non-positive. Since the congruent transformation preserves definiteness of a symmetric real matrix, the diagonal elements of the damping and stiffness matrices are always nonnegative[14].

The last component of the transformation is the square root of the final mass matrix M_f. As will be shown, if an orthogonal matrix can be found such that the realizability conditions mentioned above are satisfied, the final mass matrix can be computed explicitly.

Given the decomposition of the transformation in (16), obtaining realizable form reduces to solving for an appropriate orthogonal matrix R. Orthogonal matrices are comprised of rotations, with determinant $+1$, and reflections, with determinant -1. In addition, permutation matrices constitute a subset of both rotation and reflection matrices. Used in a congruence transformation, permutation matrices simply reorder the coordinates.

A basis for orthogonal matrices can be constructed from the rotation matrices plus a single arbitrary reflection. Choosing this reflection as a permutation matrix reduces the basis, without loss of generality, to the rotation matrices. The $n \times n$ rotation matrices constitute the Special Orthogonal group, $SO(n)$. In the remainder of the paper, rotation matrices will be used as a basis for R.

B. Parameterization of the Orthogonal Transformation

The component R of the transformation (16) must perform two tasks, aligning the input and output influence matrices as well as ensuring that the off-diagonal elements of the mass-normalized stiffness and damping matrices are nonpositive. These tasks can be performed sequentially by writing R as the product of two rotation matrices

$$
R = R_i R_o \tag{17}
$$

where the component R_i aligns the influence matrices. R_o ensures nonpositive off-diagonal elements of the stiffness and damping matrices while preserving the form of the influence matrices obtained with R_i.

1. *Aligning Input and Output Influence Matrices*

For the first task, denote the coordinate transformation as

$$
x = M_0^{-1/2} R_i \tilde{z} \tag{18}
$$

Substituting (18) into the initial model (11) and premultiplying by $R_i^T M_0^{-1/2}$ yields the following model

$$
\begin{aligned}
\ddot{\tilde{z}} + C_{\tilde{z}}\dot{\tilde{z}} + K_{\tilde{z}}\tilde{z} &= F_{\tilde{z}} u \\
y &= H_{\tilde{z}}\ddot{\tilde{z}}
\end{aligned}
\tag{19}
$$

in which the matrices are defined by

$$
\begin{aligned}
C_{\tilde{z}} &= R_i^T M_0^{-1/2} C_0 M_0^{-1/2} R_i \\
K_{\tilde{z}} &= R_i^T M_0^{-1/2} K_0 M_0^{-1/2} R_i \\
F_{\tilde{z}} &= R_i^T F_z \\
H_{\tilde{z}} &= H_z R_i
\end{aligned} \tag{20}
$$

where $F_z = M_0^{-1/2} F_0$ and $H_z = H_0 M_0^{-1/2}$.

R_i can be obtained by QR factorization of the mass normalized input and output influence matrices, F_z and H_z. In this QR factorization, a matrix is decomposed into a product of an orthogonal matrix and an upper triangular matrix. A property of this method is that a matrix whose column vectors are perpendicular to each other can be factored as a product of an orthogonal matrix and a diagonal matrix.

In the most general case, the input and output influence matrices in the final realizable model (13) must have the form given by (10). With consideration of (16), it can be proved that the columns of F_z are mutually orthogonal. Thus, after the rotation R_i, the input and output influence matrices $F_{\tilde{z}}$ and $H_{\tilde{z}}$ in (19) should satisfy the following relationships

$$
F_{\tilde{z}} = R_i^T F_z = \left[\ ||f_{z_1}||e_1 \ \ ||f_{z_2}||e_2 \ \ \cdots \ \ ||f_{z_m}||e_m \ \right]
$$

$$
H_{\tilde{z}} = H_0 M_0^{-1/2} R_i = \begin{bmatrix} ||f_{z_1}||e_1^T \\ ||f_{z_2}||e_2^T \\ \vdots \\ ||f_{z_r}||e_r^T \\ ||h_{z_{(r+1)}}||e_{m+1}^T \\ ||h_{z_{(r+2)}}||e_{m+2}^T \\ \cdots \\ ||h_{z_p}||e_{m+(p-r)}^T \end{bmatrix} \tag{21}
$$

To fulfill these requirements, the component R_i can be decomposed as a product of two rotations

$$
R_i = R_{F_z} R_{H_z} \tag{22}
$$

In (22), the first component R_{F_z} satisfies

$$
R_{F_z}^T F_z = \left[\ ||f_{z_1}||e_1 \ \ ||f_{z_2}||e_2 \ \ \cdots \ \ ||f_{z_m}||e_m \ \right] \tag{23}
$$

Suppose the QR factorization of F_z is given by

$$
Q_{F_z} \left[\ ||f_{z_1}||e_1 \ \ ||f_{z_2}||e_2 \ \ \cdots \ \ ||f_{z_m}||e_m \ \right] = F_z \tag{24}
$$

where Q_{F_z} is an orthogonal matrix. The first component R_{F_z} then can be chosen as

$$
R_{F_z} = Q_{F_z} \tag{25}
$$

From (23), the first m column vectors of R_{F_z} (or Q_{F_z}) should be equal to $f_{z_i}/||f_{z_i}||$ $(i = 1, 2, \cdots, m)$, respectively. According to (10), $H_z R_{F_z}$ should have the following form

$$
H_z R_{F_z} = \begin{bmatrix} ||f_{z_1}||e_1^T \\ ||f_{z_2}||e_2^T \\ \vdots \\ ||f_{z_r}||e_r^T \\ \bar{H}_z \end{bmatrix} \tag{26}
$$

where $\bar{H}_z = \left[\ 0_{(p-r)\times m} \ \ \tilde{H}_z \ \right]$ and $\tilde{H}_z$ is a $(p-r)\times(n-m)$ matrix.

The second component R_{H_z} of the transformation R_i needs to preserve e_j's $(j = 1, 2, \cdots, m)$ and should convert (26) to the the following form

$$
(H_z R_{F_z}) R_{H_z} = \begin{bmatrix} ||f_{z_1}||e_1^T \\ ||f_{z_2}||e_2^T \\ \vdots \\ ||f_{z_r}||e_r^T \\ ||h_{z_{(r+1)}}||e_{m+1}^T \\ ||h_{z_{(r+2)}}||e_{m+2}^T \\ \cdots \\ ||h_{z_p}||e_{m+(p-r)}^T \end{bmatrix} \tag{27}
$$

Suppose the QR factorization of $\tilde{H}_z^T$ is given by

$$
Q_{H_z} \left[\ ||h_{z_{(r+1)}}||\tilde{e}_1 \ \ ||h_{z_{(r+2)}}||\tilde{e}_2 \ \ \cdots \ \ ||h_{z_p}||\tilde{e}_{(p-r)} \ \right] = \tilde{H}_z^T \tag{28}
$$

where Q_{H_z} is an orthogonal matrix and $\tilde{e}_i$ is an element of the standard basis for $\Re^{n-m}$, which has a 1 at its $i'th$ component and 0's elsewhere. Equation (28) can be rewritten as

$$
\begin{bmatrix} ||h_{z_{(r+1)}}||\tilde{e}_1^T \\ ||h_{z_{(r+2)}}||\tilde{e}_2^T \\ \vdots \\ ||h_{z_p}||\tilde{e}_{(p-r)}^T \end{bmatrix} Q_{H_z}^T = \tilde{H}_z \tag{29}
$$

or equivalently

$$
\begin{bmatrix} ||h_{z_{(r+1)}}||\tilde{e}_1^T \\ ||h_{z_{(r+2)}}||\tilde{e}_2^T \\ \vdots \\ ||h_{z_p}||\tilde{e}_{(p-r)}^T \end{bmatrix} = \tilde{H}_z Q_{H_z} \tag{30}
$$

Thus, the second component R_{H_z} in (22) is given by

$$
R_{H_z} = \begin{bmatrix} I_{m\times m} & 0 \\ 0 & Q_{H_z} \end{bmatrix} \tag{31}
$$

In summary, from (22), (25) and (31), the component R_i of the transformation R in (17) is given by

$$
R_i = R_{F_z} R_{H_z} = Q_{F_z} \begin{bmatrix} I_{m\times m} & 0 \\ 0 & Q_{H_z} \end{bmatrix} \tag{32}
$$

2. *Achieving Nonpositive Offdiagonal Damping and Stiffness Elements*

After aligning the input and output influence matrices, a second rotation R_o is needed to convert the damping and stiffness matrices in (19) to mechanically realizable form in which all off-diagonal elements are nonpositive. An explicit solution for R_o is not available, however, its form and the number of its free parameters can be derived as follows. Denote the coordinate transformation

$$\tilde{z} = R_o w \tag{33}$$

Substituting this transformation into (19) and pre-multiplying by R_o^T yields the following second order model

$$\begin{aligned} \ddot{w} + C_w \dot{w} + K_w w &= F_w u \\ y &= H_w \ddot{w} \end{aligned} \tag{34}$$

in which

$$\begin{aligned} C_w &= R_o^T C_{\tilde{z}} R_o \\ K_w &= R_o^T K_{\tilde{z}} R_o \\ F_w &= R_o^T F_{\tilde{z}} \\ H_w &= H_{\tilde{z}} R_o \end{aligned} \tag{35}$$

The input and output influence matrices in (19) are already in the desired form, given by (21), only with a lack of scaling, and the transformation R_o should preserve this form. To do so, it can be expressed as

$$R_o = \begin{bmatrix} I_{n_i \times n_i} & 0 \\ 0 & \tilde{R}_o \end{bmatrix}, \tag{36}$$

with $I_{n_i \times n_i}$ as the $n_i \times n_i$ identity matrix and the rotation matrix $\tilde{R}_o \in SO(n - n_i)$. The value n_i is the number of masses at which input forces are applied and / or accelerations are measured,

$$n_i = m + p - r. \tag{37}$$

It follows that $n - n_i$ is the number of internal masses of the system, i.e., those masses to which an input is not applied nor at which an output is measured.

The free parameters of R_o are those of $\tilde{R}_o \in SO(n - n_i)$, which number $(n - n_i)(n - n_i - 1)/2$. Given R_o, an explicit solution exists for the final mass matrix and so this is also the number of free parameters of the transformation space defined by (16). This number, quadratic in the number of internal masses in the model, represents the dimension of the space which must be mapped or searched for mechanically realizable models.

C. Solving for the Final Mass Matrix

An explicit solution for the final mass matrix, M_f, can be derived from the realization conditions of (5) requiring the damping and stiffness matrices in (13) to satisfy

$$\begin{aligned} K_f \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T &= 0 \\ C_f \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T &= 0. \end{aligned} \tag{38}$$

Since the model (13) is related to the model (34) by the congruent transformation $M_f^{1/2}$, $C_f = M_f^{1/2} C_w M_f^{1/2}$ and $K_f = M_f^{1/2} K_w M_f^{1/2}$. Substituting these expressions into (38) reduces to

$$\begin{aligned} C_w \sqrt{m_f} &= 0 \\ K_w \sqrt{m_f} &= 0 \end{aligned} \tag{39}$$

where $\sqrt{m_f}$ is a vector of the square roots of the final masses, i.e., $\sqrt{m_f} = \begin{bmatrix} \sqrt{m_{f_1}} & \sqrt{m_{f_2}} & \cdots & \sqrt{m_{f_n}} \end{bmatrix}^T$.

The vector $\sqrt{m_f}$ is a scaled version of the shared nullspace vector of C_w and K_w. The final masses are obtained by scaling the nullspace vector according to the following theorem, presented for the most general case of input and output influence matrices (MIMO drive-point and transfer accelerance) given by (10).

Theorem 1. *The input masses, to which excitation forces are applied, and the output masses, at which accelerations are measured, are given by*

$$\begin{aligned} m_{f_i} &= \frac{1}{||f_{z_i}||^2}, \ i = 1, 2, \cdots, m \\ m_{f_{(m+j)}} &= \frac{1}{||h_{z_{(r+j)}}||^2}, \ j = 1, 2, \cdots, p - r \end{aligned} \tag{40}$$

where f_{z_i} $(i = 1, 2, \cdots, m)$ is the i'th column vector of $M_0^{-1/2} F_0$ and h_{z_j} $(j = 1, 2, \cdots, p)$ is the j'th row vector of $H_0 M_0^{-1/2}$.

Proof. According to (10), (14) and (16),

$$F_f = T^T F_0 = M_f^{1/2} R^T M_0^{-1/2} F_0 = M_f^{1/2} R^T F_z \tag{41}$$

This is equivalent to

$$M_f^{-1/2} \begin{bmatrix} e_1 & e_2 & \cdots e_m \end{bmatrix} = R^T \begin{bmatrix} f_{z_1} & f_{z_2} & \cdots & f_{z_m} \end{bmatrix} \tag{42}$$

which simplifies to

$$\begin{bmatrix} \frac{1}{\sqrt{m_{f_1}}} e_1 & \frac{1}{\sqrt{m_{f_2}}} e_2 & \cdots & \frac{1}{\sqrt{m_{f_m}}} e_m \end{bmatrix} = \begin{bmatrix} R^T f_{z_1} & R^T f_{z_2} & \cdots & R^T f_{z_m} \end{bmatrix} \tag{43}$$

Since rotation matrices preserve vector length, equating the magnitude of columns yields

$$m_{f_i} = \frac{1}{||f_{z_i}||^2}, \ \ i = 1, 2, \cdots, m \tag{44}$$

The second equation of (40) follows similarly. □

This theorem states that each member of the set of mechanically realizable models which are input-output equivalent to the initial model (11) has the same input and output masses. The following theorem proves the invariance of total system mass for all mechanical realizations. Physically, this result follows from input-output equivalence at zero frequency to preserve the rigid body mode.

Theorem 2. *All mechanical realizations which are input-output equivalent to the original second-order model (11) possess the same total mass.*

Proof. Recall from (15) the necessary condition for realizability that C_0 and K_0 share the same nullspace vector and let this vector v_0 be of unit length:

$$C_0 v_0 = K_0 v_0 = 0 \tag{45}$$

By (5) and (14), $T^T K_0 T \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T = 0$ and since T^T is invertible, $T \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T = \alpha v_0$, where α is a scalar constant.

Substituting $T = M_0^{-1/2} R M_f^{1/2}$ yields

$$\sqrt{m_f} = \alpha R^T M_0^{1/2} v_0 \tag{46}$$

and an expression for total mass is given by

$$\sqrt{m_f}^T \sqrt{m_f} = \sum_{i=1}^{i=n} m_{f_i} = \alpha^2 v_0^T M_0 v_0 \tag{47}$$

To compute α, it is known from Theorem 1 that $\sqrt{m_{f_1}} = 1/||f_{z_1}||$. Since a system must have at least one input and output, the first mass can always be used in this expression and combining it with (46) yields

$$\alpha = \frac{1/||f_{z_1}||}{(R^T M_0^{1/2} v_0)_1} \tag{48}$$

where the subscript 1 in the denominator indicates the first element of the column vector.

Recall (17) in which R_i aligns the inputs and outputs and R_o has the structure of (36). Since all mechanical realizations share the same R_i and furthermore, since R_o cannot change the first element of $M_0^{1/2} v_0$, the constant α is independent of R_o and thus the same for all mechanical realizations. □

Taken together the preceding theorems indicate that only the internal masses of the system can differ between realizations and that the total internal mass is constant.

IV. OBTAINING REALIZABLE MODELS

In the preceding section, it has been demonstrated that congruent coordinate transformations T for converting a model to mechanically realizable form can be expressed as

$$T = M_0^{-1/2} R M_f^{1/2} = M_0^{-1/2} (R_i R_o) M_f^{1/2}. \tag{49}$$

The first component $M_0^{-1/2}$ is known from the initial second order model (11) and the second component R_i can be obtained via QR factorization of the input and output influence matrices in (19). The last component $M_f^{1/2}$ can be obtained from (39) and Theorem 1.

An explicit solution for the remaining component R_o is only available for SISO systems with no damping or proportional damping. In all other cases, a solution for R_o must be sought through mapping or selectively searching the special orthogonal group $SO(n - n_i)$ in which $n - n_i$ is the number of internal masses. $SO(n - n_i)$ can be described by n_p parameters where

$$n_p = (n - n_i)(n - n_i - 1)/2 \tag{50}$$

and each parameter corresponds to a two-dimensional rotation angle.

These parameters must be selected to satisfy the $n(n-1)$ inequality constraints that the off-diagonal components of the stiffness and damping matrices be nonpositive. Since the number of constraints exceeds the number of parameters in (50), it is not clear that a solution will exist in the general case. If the initial model is derived from either experiment or FEM, however, it is likely that these constraints will be dependent and mechanically realizable solutions will exist.

To search for a solution, R_o in (36) can be written as the product of n_p two-dimensional rotation matrices involving the last $n - n_i$ coordinates,

$$R_o = \prod_{\substack{i = n_i + 1, \\ j = n_i + 2}}^{\substack{i = n - 1, \\ j = n}} R_{ij}, \tag{51}$$

in which R_{ij} is the two-dimensional rotation matrix in the i'th and j'th coordinates. Two-dimensional rotations, also known as Givens rotations, have been widely used to convert symmetric matrices to tridiagonal matrices in solving symmetric matrix eigenvalue problems[15].

The elements of these rotation matrices correspond to those of an identity matrix except for the following four.

$$\begin{aligned} R_{ij}(i,i) &= \cos(\theta_{ij}), \quad R_{ij}(i,j) = -\sin(\theta_{ij}) \\ R_{ij}(j,i) &= \sin(\theta_{ij}), \quad R_{ij}(j,j) = \cos(\theta_{ij}) \end{aligned} \tag{52}$$

To obtain a bijection (one-to-one and onto map) between θ_{ij} and $SO(n - n_i)$ where $n - n_i \geq 3$, it is not necessary for all θ_{ij} to vary as $0 \leq \theta_{ij} < 2\pi$. For example, in $SO(3)$, all rotation matrices can be generated from the product $R_{12}(\theta_{12})R_{13}(\theta_{13})R_{23}(\theta_{23})$ in which $0 \leq \theta_{12} < 2\pi$, $0 \leq \theta_{13} < \pi$ and $0 \leq \theta_{23} < 2\pi$. Allowing $0 \leq \theta_{13} < 2\pi$ would result in a two-to-one map.

Even when the angle ranges are appropriately restricted so that the map from θ_{ij} to $SO(n - n_i)$ is a bijection, the map from $SO(n - n_i)$ to the system model (34) is many-to-one. This is due to the equivalence class of models corresponding to permutations of the internal masses.

Recall that a permutation matrix congruence transformation swaps pairs of rows and columns of the matrix to which it is applied. This operation results purely in a renumbering of the internal mass coordinates of the model. There are $(n - n_i)!$ possible permutations of the internal masses. Half of these correspond to swapping an even number of pairs of rows and columns and so result from rotation permutation matrices. As a result, the mapping from θ_{ij} to the system model (34) will be $(n - n_i)!/2$-to-one.

The following sections describe how realizable models can be found by mapping or selectively searching the space of transformations. For each example, the initial second order model was generated by applying a random congruent transformation to a realizable second order model. The first two examples involve mapping the entire transformation space and so the initial models are recovered as members of the sets of realizable models.

A. Mapping Transformation Space

When the number of internal masses is small, the number of free parameters of the transformation space, given by n_p in (50), is also small. In this case, a complete mapping of transformation space is feasible and the results can be easily visualized. Two examples with three internal masses are presented here.

Example 1: Four-mass Driving-point Accelerance

An initial second order model satisfying (15) is given by

$$\begin{bmatrix} 1.5182 & 3.8080 & -1.0679 & 1.8792 \\ 3.8080 & 10.4989 & -2.9426 & 5.2055 \\ -1.0679 & -2.9426 & 0.9188 & -1.4534 \\ 1.8792 & 5.2055 & -1.4534 & 2.7745 \end{bmatrix} \ddot{x} +$$
$$\begin{bmatrix} 0.1836 & 0.3089 & 0.0234 & 0.0841 \\ 0.3089 & 1.1514 & -0.0858 & 0.4348 \\ 0.0234 & -0.0858 & 0.0728 & 0.0187 \\ 0.0841 & 0.4348 & 0.0187 & 0.2717 \end{bmatrix} \dot{x} +$$
$$\begin{bmatrix} 6818 & 16814 & -2200 & 2656 \\ 16814 & 63328 & -11552 & 14063 \\ -2200 & -11552 & 3827 & -910 \\ 2656 & 14063 & -910 & 6627 \end{bmatrix} x = \begin{bmatrix} -0.4326 \\ -1.1465 \\ 0.3273 \\ -0.5883 \end{bmatrix} u$$
$$y = \begin{bmatrix} -0.4326 & -1.1465 & 0.3273 & -0.5883 \end{bmatrix} \ddot{x} \quad (53)$$

After mass normalization, the initial model becomes

$$\ddot{w} + \begin{bmatrix} 0.7316 & -0.1283 & 0.3312 & -0.1787 \\ -0.1283 & 0.4246 & 0.2858 & -0.1413 \\ 0.3312 & 0.2858 & 1.0947 & 0.2789 \\ -0.1787 & -0.1413 & 0.2789 & 0.5134 \end{bmatrix} \dot{w} +$$
$$\begin{bmatrix} 17792 & -6182 & 5980 & -2606 \\ -6182 & 27522 & 6075 & -20547 \\ 5980 & 6075 & 21560 & 05432 \\ -2606 & -20547 & 5432 & 28812 \end{bmatrix} w = \begin{bmatrix} -0.1401 \\ -0.2679 \\ 0.0935 \\ -0.1728 \end{bmatrix} u$$
$$y = \begin{bmatrix} -0.1401 & -0.2679 & 0.0935 & -0.1728 \end{bmatrix} \ddot{w} \quad (54)$$

By QR factorization of the vector $\begin{bmatrix} -0.1401 & -0.2679 & 0.0935 & -0.1728 \end{bmatrix}^T$, the orthogonal matrix R_i is obtained as

$$R_i = \begin{bmatrix} -0.3886 & -0.7430 & 0.2592 & -0.4793 \\ -0.7430 & 0.6025 & 0.1387 & -0.2565 \\ 0.2592 & 0.1387 & 0.9516 & 0.0895 \\ -0.4793 & -0.2565 & 0.0895 & 0.8345 \end{bmatrix} \quad (55)$$

After aligning the input and output influence vectors, the mass normalized second order model (54) is

$$\ddot{\tilde{z}} + \begin{bmatrix} 0.0490 & -0.0882 & -0.2034 & 0.0236 \\ -0.0882 & 0.6629 & -0.0759 & 0.1068 \\ -0.2034 & -0.0759 & 1.3181 & 0.0127 \\ 0.0236 & 0.1068 & 0.0127 & 0.7344 \end{bmatrix} \dot{\tilde{z}} +$$
$$\begin{bmatrix} 0.1877 & -0.5873 & -0.4114 & 0.4359 \\ -0.5873 & 3.2409 & -0.0973 & -1.3774 \\ -0.4114 & -0.0973 & 2.5883 & -0.0377 \\ 0.4359 & -1.3774 & -0.0377 & 3.5517 \end{bmatrix} \tilde{z} = \begin{bmatrix} 0.3605 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0.3605 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \ddot{\tilde{z}} \quad (56)$$

The number of internal masses is $n - n_i = 3$ and so the transformation space can be parameterized by $n_p = 3$ two-dimensional rotations. The terms R_{ij} in (51) which preserve the driving-point input and output influence vectors are given by

$$R_{23} = \begin{bmatrix} 1 & & & \\ & cos\theta_{23} & -sin\theta_{23} & \\ & sin\theta_{23} & cos\theta_{23} & \\ & & & 1 \end{bmatrix}$$
$$R_{24} = \begin{bmatrix} 1 & & & \\ & cos\theta_{24} & & -sin\theta_{24} \\ & & 1 & \\ & sin\theta_{24} & & cos\theta_{24} \end{bmatrix}$$
$$R_{34} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & cos\theta_{34} & -sin\theta_{34} \\ & & sin\theta_{34} & cos\theta_{34} \end{bmatrix} \quad (57)$$

The matrix R_o is the product

$$R_o(\theta_{23}, \theta_{24}, \theta_{34}) = R_{23}R_{24}R_{34}. \quad (58)$$

A complete map relating rotation angles to realizable models is obtained by discretizing the rotation angles as shown in Figure 2. The shaded regions correspond to mechanically realizable models. Note that the plotted angle ranges are $0 \le \theta_{23} < 2\pi$, $\pi/2 \le \theta_{24} < 3\pi/2$, $0 \le \theta_{34} < 2\pi$ in order to obtain one-to-one coverage of $SO(3)$. Since there are three internal masses, there are six possible permutations of these masses, three of which are obtained through rotations. Consequently, the map from θ_{ij} to system models is three-to-one resulting in three equivalent regions of mechanically realizable models. Removing equivalent realizations reduces the set to that shown in Figure 3.

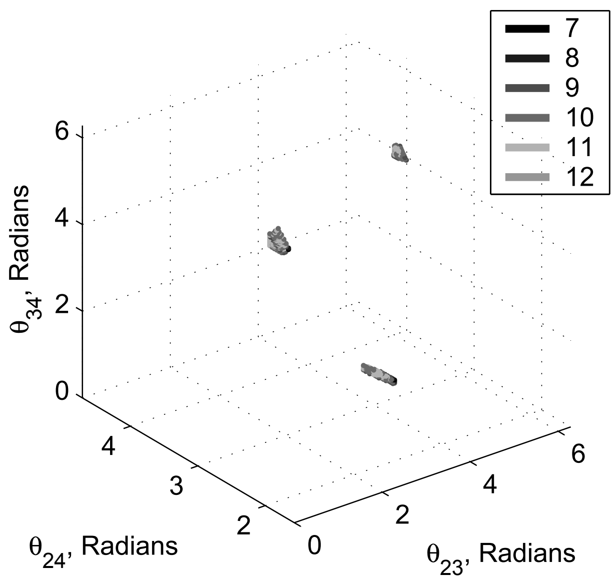

FIG. 2: Realizable Regions for Example 1. Three regions correspond to cyclic permutations of the three internal masses. Legend indicates total number of connecting elements (springs and dampers) in realizations.

In the figures, the shading indicates the number of connecting elements (springs and dampers) in the realizations. The realizations with the fewest connecting elements are located on the boundary between realizable and unrealizable regions where off-diagonal elements of the damping and stiffness matrices change their signs.

Two realizable models from this set are presented here which differ in the number of springs and dampers.

Realization 1: Selection of rotation angles $\theta_{23} = 2.9496$, $\theta_{24} = 2.3387$ and $\theta_{34} = 1.7104$ radians yields

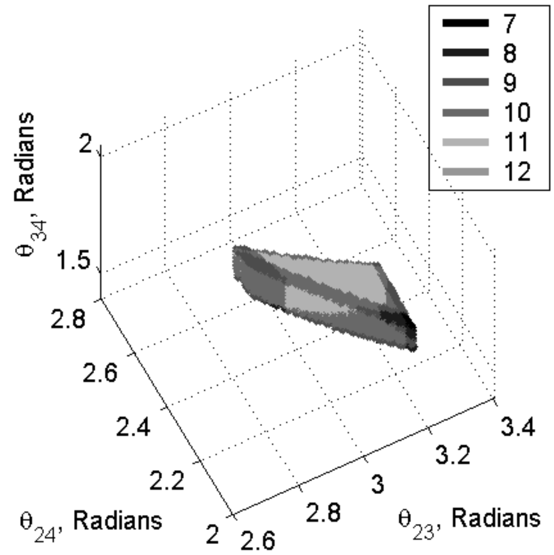

FIG. 3: Single Region Containing all Distinct Mechanical Realizations for Example 1. Legend indicates total number of connecting elements (springs and dampers) in realizations.

the mechanically realizable model

$$
\begin{bmatrix} 7.6941 & & & \\ & 0.0164 & & \\ & & 0.1906 & \\ & & & 0.2268 \end{bmatrix} \ddot{q}
+ \begin{bmatrix} 0.3770 & -0.0058 & -0.0974 & -0.2738 \\ -0.0058 & 0.0136 & -0.0012 & -0.0066 \\ -0.0974 & -0.0012 & 0.1124 & -0.0139 \\ -0.2738 & -0.0066 & -0.0139 & 0.2943 \end{bmatrix} \dot{q}
+ \begin{bmatrix} 14444 & -115 & -8796 & -5533 \\ -115 & 339 & -124 & -100 \\ -8796 & -124 & 9078 & -158 \\ -5533 & -100 & -158 & 5792 \end{bmatrix} q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u
$$
$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \ddot{q} \quad (59)
$$

Since there are no zero elements in the damping and stiffness matrices, this realization includes a dashpot and spring between each pair of masses.

Realization 2: Rotation angles $\theta_{23} = 3.2484$, $\theta_{24} = 2.1776$ and $\theta_{34} = 1.5769$ radians produce a mechanically realizable model with the fewest springs (four) and dashpots (three), as shown in Figure 4.

$$
\begin{bmatrix} 7.6941 & 0 & 0 & 0 \\ 0 & 0.0220 & 0 & 0 \\ 0 & 0 & 0.2502 & 0 \\ 0 & 0 & 0 & 0.1616 \end{bmatrix} \ddot{x} +
$$

$$
\begin{bmatrix} 0.3770 & -0.0177 & -0.1450 & -0.2143 \\ -0.0177 & 0.0178 & 0 & -0.0001 \\ -0.1450 & 0 & 0.1450 & 0 \\ -0.2143 & -0.0001 & 0 & 0.2144 \end{bmatrix} \dot{x}
$$

$$
+ \begin{bmatrix} 14444 & 0 & -10633 & -3810 \\ 0 & 474 & -474 & 0 \\ -10633 & -474 & 11528 & -421 \\ -3810 & 0 & -421 & 4232 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \ddot{x} \quad (60)
$$

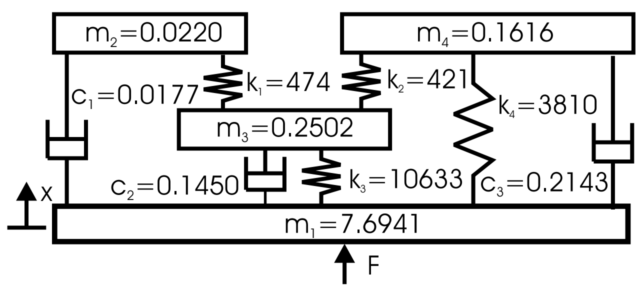

FIG. 4: Realization with the Fewest Springs and Dashpots.

Although the initial model (53) and two realizations (59) and (60) have different mass, damping and stiffness matrices, they possess the same driving-point accelerance.

Example 2: Five-mass Transfer Accelerance

Consider the SISO second order model satisfying (15) and given by

$$
\begin{bmatrix} 36.0632 & 0.2743 & 7.0077 & 37.1531 & -14.0227 \\ 0.2743 & 16.5796 & -6.7431 & 5.5551 & 2.1107 \\ 7.0077 & -6.7431 & 17.1533 & 16.6964 & -13.3775 \\ 37.1531 & 5.5551 & 16.6964 & 61.9214 & -25.8255 \\ -14.0227 & 2.1107 & -13.3775 & -25.8255 & 17.8591 \end{bmatrix} \ddot{x} +
$$

$$
\begin{bmatrix} 97.2937 & -57.5426 & 91.8234 & 72.4757 & -38.5575 \\ -57.5426 & 62.9875 & -56.3684 & -51.0698 & 29.9845 \\ 91.8234 & -56.3684 & 92.8620 & 64.1156 & -37.4205 \\ 72.4757 & -51.0698 & 64.1156 & 60.7210 & -27.3858 \\ -38.5575 & 29.9845 & -37.4205 & -27.3858 & 34.8577 \end{bmatrix} \dot{x} +
$$

$$
\begin{bmatrix} 931.9559 & -207.5005 & 567.4285 & 768.4827 & -618.2834 \\ -207.5005 & 383.5154 & -262.6956 & -186.1123 & 172.2111 \\ 567.4285 & -262.6956 & 514.7016 & 383.3350 & -393.4070 \\ 768.4827 & -186.1123 & 383.3350 & 709.9495 & -487.2931 \\ -618.2834 & 172.2111 & -393.4070 & -487.2931 & 544.2288 \end{bmatrix} x
$$

$$
= \begin{bmatrix} 1.9574 & -0.2111 & 0.5512 & 0.4620 & -1.2316 \end{bmatrix}^T u
$$

$$
y = \begin{bmatrix} 0.5045 & 1.1902 & -1.0998 & -0.3210 & 1.0556 \end{bmatrix} \ddot{x} \quad (61)
$$

The input and output influence vectors differ indicating that the system represents a transfer accelerance. Following (7), a realizable model is sought in which the force excitation is applied at the first coordinate and the acceleration is measured at the second. The solution for R_i is not included here for the sake of brevity.

With $n = 5$ masses and $n_i = 2$ input and output masses, the transformation space is parameterized by $n_p = 3$ two-dimensional rotations. The matrix R_o is given by

$$
R_o(\theta_{34}, \theta_{35}, \theta_{45}) = R_{34} R_{35} R_{45}. \quad (62)
$$

Figure 5 depicts the map between rotation angles and system models. As in Example 1, there are three equivalent regions of mechanically realizable models corresponding to rotational permutations of the three internal masses. The plotted angle ranges in this figure are $0 \leq \theta_{34} < 2\pi$, $0 \leq \theta_{35} < \pi$, $0 \leq \theta_{45} < 2\pi$.

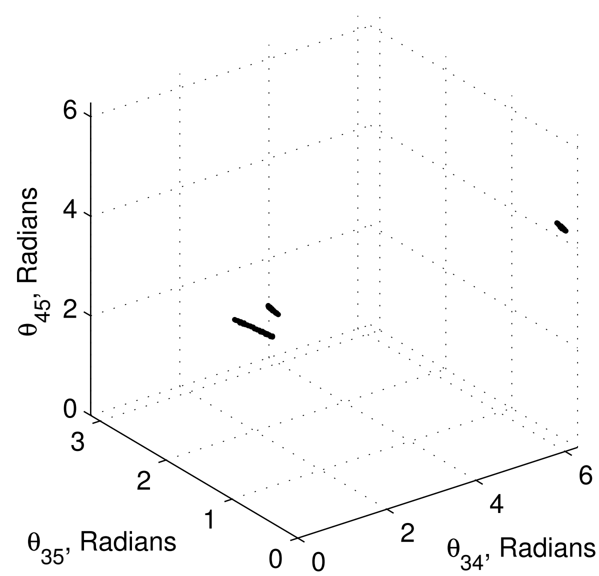

FIG. 5: Realizable Regions for Example 2. Three regions correspond to cyclic permutations of the three internal masses.

As an example realization, rotation angles $\theta_{34} = 3.0386$, $\theta_{35} = 3.0048$ and $\theta_{45} = 1.1158$ radians produce the following mechanically realizable model with a fully populated stiffness matrix and a damping matrix possessing a single zero dashpot.

$$
\begin{aligned}
&\begin{bmatrix} 2.0000 & 0 & 0 & 0 & 0 \\ 0 & 5.0000 & 0 & 0 & 0 \\ 0 & 0 & 3.4340 & 0 & 0 \\ 0 & 0 & 0 & 2.3326 & 0 \\ 0 & 0 & 0 & 0 & 10.2334 \end{bmatrix} \ddot{q} + \\
&\begin{bmatrix} 8.5000 & -1.0000 & -6.8047 & -0.0596 & -0.6357 \\ -1.0000 & 13.3000 & -11.0845 & -0.6876 & -0.5280 \\ -6.8047 & -11.0845 & 31.9340 & -14.0385 & -0.0063 \\ -0.0596 & -0.6876 & -14.0385 & 14.7857 & -0.0000 \\ -0.6357 & -0.5280 & -0.0063 & -0.0000 & 1.1700 \end{bmatrix} \dot{q} + \\
&\begin{bmatrix} 130.0000 & -20.0000 & -49.9987 & -8.2885 & -51.7128 \\ -20.0000 & 92.0000 & -47.6153 & -20.9996 & -3.3852 \\ -49.9987 & -47.6153 & 139.9326 & -29.8486 & -12.4700 \\ -8.2885 & -20.9996 & -29.8486 & 59.6030 & -0.4664 \\ -51.7128 & -3.3852 & -12.4700 & -0.4664 & 68.0344 \end{bmatrix} q \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T u \\
y &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \ddot{q}
\end{aligned}
\tag{63}
$$

B. Searching Transformation Space

As the number of internal masses $n - n_i$ in the model grows, it becomes impractical to map the entire transformation space as was done in the preceding examples. Instead, the transformation space can be selectively searched using a nonlinear optimization method.

Recalling that the role of R_o in obtaining a realizable model is to ensure that the off-diagonal elements of the stiffness and damping matrices are nonpositive, a cost function for optimization can be chosen as

$$J(\theta) = w_1 S_K + w_2 S_C \tag{64}$$

where θ is the vector of rotation angles, S_K is the summation of all positive off-diagonal elements in the stiffness matrix K and S_C is the summation of all positive off-diagonal elements in the damping matrix C. In order to balance the contributions from the stiffness and damping matrices, the weighting factors w_1 and w_2 are defined as

$$
\begin{aligned}
w_1 &= 1 \\
w_2 &= \frac{trace(K)}{trace(C)}.
\end{aligned}
$$

Since a congruent orthogonal transformation does not change the trace of a matrix, the weighting factor w_2 is constant.

A wide variety of optimization techniques can be employed to search for an angle vector θ resulting in a realizable model. Since the problem is nonlinear, local minima of the cost function can exist. If such a minima is detected during optimization, a perturbation of random direction and magnitude can be applied to escape its domain of attraction.

Example 3: Ten-mass SISO Driving-point Accelerance

To illustrate the use of an optimization method in solving the mechanical realization problem, the Nelder-Mead method[16] was applied to the following ten-mass driving-point system using the cost function defined in (64). For brevity, the model is presented after mass normalization and alignment of input and output influence vectors. The damping matrix does not correspond to proportional damping.

$$
C_{\tilde{z}} = 10^{-2} \times \begin{bmatrix}
2.71 & 0.59 & 0.91 & 0.31 & 0.28 & 0.04 & -0.53 & 0.38 & -0.86 & 0.10 \\
0.59 & 2.21 & -0.45 & 0.18 & -0.56 & -0.00 & 0.14 & 0.33 & 0.74 & -0.21 \\
0.91 & -0.45 & 3.24 & 0.10 & -0.72 & 0.12 & 0.23 & 0.54 & 0.83 & 0.14 \\
0.31 & 0.18 & 0.10 & 2.48 & -0.02 & -0.54 & -0.12 & -0.87 & 0.86 & -0.13 \\
0.28 & -0.56 & -0.72 & -0.02 & 2.51 & -0.53 & -0.00 & 0.32 & -0.16 & -0.01 \\
0.04 & -0.00 & 0.12 & -0.54 & -0.53 & 3.21 & -0.33 & 0.07 & 0.01 & -0.07 \\
-0.53 & 0.14 & 0.23 & -0.12 & -0.00 & -0.33 & 2.91 & -0.31 & -0.23 & 0.18 \\
0.38 & 0.33 & 0.54 & -0.87 & 0.32 & 0.07 & -0.31 & 2.74 & 0.51 & -0.68 \\
-0.86 & 0.74 & 0.83 & 0.86 & -0.16 & 0.01 & -0.23 & 0.51 & 2.60 & 0.59 \\
0.10 & -0.21 & 0.14 & -0.13 & -0.01 & -0.07 & 0.18 & -0.68 & 0.59 & 3.54
\end{bmatrix}
$$

$$
K_{\tilde{z}} = \begin{bmatrix}
16685 & 5279 & 2255 & 964 & 2052 & 1663 & -3627 & 432 & -6797 & 93 \\
5279 & 11712 & -615 & -742 & 952 & 664 & -1747 & -2261 & 3368 & -2184 \\
2255 & -615 & 13728 & -1508 & -3500 & 709 & 2709 & -669 & 3985 & 1844 \\
964 & -742 & -1508 & 15194 & -821 & -457 & 360 & -362 & 4447 & 249 \\
2052 & 952 & -3500 & -821 & 17704 & -591 & -855 & -525 & 2505 & -2485 \\
1663 & 664 & 709 & -457 & -591 & 19298 & -199 & 1753 & 1535 & 34 \\
-3627 & -1747 & 2709 & 360 & -855 & -199 & 13734 & -1139 & -153 & 2045 \\
432 & -2261 & -669 & -362 & -525 & 1753 & -1139 & 16008 & 808 & -1654 \\
-6797 & 3368 & 3985 & 4447 & 2505 & 1535 & -153 & 808 & 15770 & 3236 \\
93 & -2184 & 1844 & 249 & -2485 & 34 & 2045 & -1654 & 3236 & 19318
\end{bmatrix}
$$

$$F_{\tilde{z}} = H_{\tilde{z}}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{65}$$

With nine internal masses, there are thirty-six two-dimensional rotation parameters describing the space of transformations. The optimization method was initiated with the rotation angles set to random numbers in the range 0 to 2π. The search terminates when a realizable model is found. The result of one trial appears below. This trial involved six iterations in which at most 2500 evaluations of the cost function were permitted for each iteration.

$$M_f = diag(\begin{bmatrix} 1.000 & 0.590 & 0.221 & 0.302 & 0.461 & 0.310 & 0.378 & 0.123 & 0.378 & 0.521 \end{bmatrix})$$

$$
C_f = 10^{-2} \times \begin{bmatrix}
2.71 & -0.49 & -0.05 & -0.18 & -0.60 & -0.24 & -0.29 & -0.07 & -0.45 & -0.36 \\
-0.49 & 1.84 & -0.00 & -0.07 & -0.56 & -0.02 & -0.29 & -0.02 & -0.07 & -0.34 \\
-0.05 & -0.00 & 0.77 & -0.13 & -0.07 & -0.12 & -0.05 & -0.00 & -0.18 & -0.16 \\
-0.18 & -0.07 & -0.13 & 0.88 & -0.06 & -0.16 & -0.06 & -0.00 & -0.12 & -0.09 \\
-0.60 & -0.56 & -0.07 & -0.06 & 1.45 & -0.00 & -0.00 & -0.01 & -0.04 & -0.10 \\
-0.24 & -0.02 & -0.12 & -0.16 & -0.00 & 0.88 & -0.21 & -0.00 & -0.10 & -0.03 \\
-0.29 & -0.29 & -0.05 & -0.06 & -0.00 & -0.21 & 1.27 & -0.00 & -0.22 & -0.15 \\
-0.07 & -0.02 & -0.00 & -0.00 & -0.01 & -0.00 & -0.00 & 0.10 & -0.00 & -0.00 \\
-0.45 & -0.07 & -0.18 & -0.12 & -0.04 & -0.10 & -0.22 & -0.00 & 1.24 & -0.06 \\
-0.36 & -0.34 & -0.16 & -0.09 & -0.10 & -0.03 & -0.15 & -0.00 & -0.06 & 1.29
\end{bmatrix}
$$

$$
K_f = \begin{bmatrix}
16685 & -4698 & -265 & -1705 & -1809 & -2239 & -1058 & -74 & -1632 & -3205 \\
-4698 & 10694 & -7 & -809 & -1432 & -702 & -2497 & -315 & -41 & -193 \\
-265 & -7 & 3609 & -754 & -129 & -278 & -135 & -422 & -718 & -900 \\
-1705 & -809 & -754 & 4805 & -295 & -759 & -4 & -68 & -407 & -4 \\
-1809 & -1432 & -129 & -295 & 6300 & -1016 & -522 & -1 & -1088 & -8 \\
-2239 & -702 & -278 & -759 & -1016 & 6269 & -876 & -359 & -8 & -33 \\
-1058 & -2497 & -135 & -4 & -522 & -876 & 6655 & -96 & -1182 & -285 \\
-74 & -315 & -422 & -68 & -1 & -359 & -96 & 1967 & -178 & -453 \\
-1632 & -41 & -718 & -407 & -1088 & -8 & -1182 & -178 & 5453 & -199 \\
-3205 & -193 & -900 & -4 & -8 & -33 & -285 & -453 & -199 & 5280
\end{bmatrix}
$$

$$F_f = H_f^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{66}$$

V. CONCLUSIONS

A congruent coordinate transformation has been developed to convert second order models to a form interpretable as a passive mechanical system. The space of transformations has been parameterized by a set of two-dimensional rotation matrices. Complete mapping of transformation space is possible for systems with small numbers of internal masses, however, an optimization method is needed to search for realizable models in higher dimensional cases. While not demonstrated here, optimization methods allow the flexibility to search for mechanically realizable models that satisfy additional criteria. For example, the cost function could be adapted to find models with the fewest dashpots or springs. These results can be applied to the vibration testing of complicated structures and to the design of electro-mechanical filters.

ACKNOWLEDGEMENTS

This work was supported by the Office of Naval Research under grants N00014-01-1-0155 and N00014-03-1-0881.

REFERENCES

[1] R. A. Johnson, "Mechanical Filters in Electronics" (John Wiley & Sons, Inc., 1983).
[2] S. Falk, "Die Abbildung eines allgemeine schwingungssystems auf eine einfache Schwingerkette", Ingenieur-Archiv, Vol. 23, pp. 314–328 (1955).
[3] S. L. Chen and M. Géradin, "An Exact Model Reduction Procedure for Mechanical Systems", Comput. Methods in Appl. Mech. and Eng., Vol. 143, pp. 69–78 (1997).
[4] G. J. O'Hara and P. F. Cunniff, "Elements of Normal Mode Theory", Naval Research Laboratory Report, 1963.
[5] A. D. Pierce, "Resonant-Frequency-Distribution of Internal Mass Inferred from Mechanical Impedance Matrices, with Application to Fuzzy Structure Theory", Trans. Am. Soc. Mech. Eng.: J. of Vib. and Acoust., Vol. 119, pp. 325–333 (1997).
[6] S. D. Garvey and M.I. Friswell and U. Prells, "Coordinate Transformations for Second Order Systems. Part I: General Transformations", J. of Sound and Vib., Vol. 258(5), pp. 885-909 (2002).
[7] S. D. Garvey and M.I. Friswell and U. Prells, "Coordinate Transformations for Second Order Systems. Part II: Elementary Structure-Preserving Transformations", J. of Sound and Vib., Vol. 258(5), pp. 911-930 (2002).
[8] G. M. L. Gladwell, "Inverse Problems in Vibration", Appl. Mech. Rev., Vol. 49, pp. S25–S34 (1996).
[9] M. T. Chu, "Inverse Eigenvalue Problems", SIAM Rev., Vol. 40, pp. 1–39 (1998).
[10] Y. M. Ram and S. Elhay, "An Inverse Eigenvalue Problem for the Symmetric Positive Linear Pencil with Applications to Vibrating Systems", Proceedings of the Computational Techniques and Applications: CTAC 97, edited by B. J. Noye and M. D. Teubner and A. W. Gill, pp. 561-568 (World Scientific Publishing Co., 1998).
[11] O. Rojo and R. Soto and J. Egana, "A Note on the Construction of a Positive Oscillatory Matrix with a Prescribed Spectrum", Comput. Math. Appl., Vol. 41, pp. 353-361 (2001).
[12] H. Baher, "Synthesis of Electrical Networks" (John Wiley & Sons, Inc., 1984).
[13] N. M. M. Maia and Silva, J. M. M. Silva, "Theoretical and Experimental Modal Analysis" (Research Studies Press LTD., Taunton, England, 1997).
[14] R. A. Horn and C. R. Johnson, "Matrix Analysis" (Cambridge University Press, Cambridge, UK, 1990).
[15] A. Jennings, "Matrix Computation" (John Wiley & Sons, 1992).
[16] W. H. Press and S. A. Teukolsky and W. T. Vetterling and B. P. Flannery, "Numerical Recipes in C: the Art of Scientific Computing" (Cambridge University Press, New York, 1992), 2nd ed.